ZOMETOOLS IN THE CLASSROOM

Contents

Introduction

The *Zometool* is used by some of the world's most renowned mathematicians and scientists, and by the world's most ordinary children. The *Zometool* company seeks to reawaken interest in geometry by allowing everyone to build vivid, memorable geometrical images that appeal to the eye, attract the heart, and sharpen the mind..

This manual contains advice and suggestions that have come from actual classroom experience, from junior high school into college.

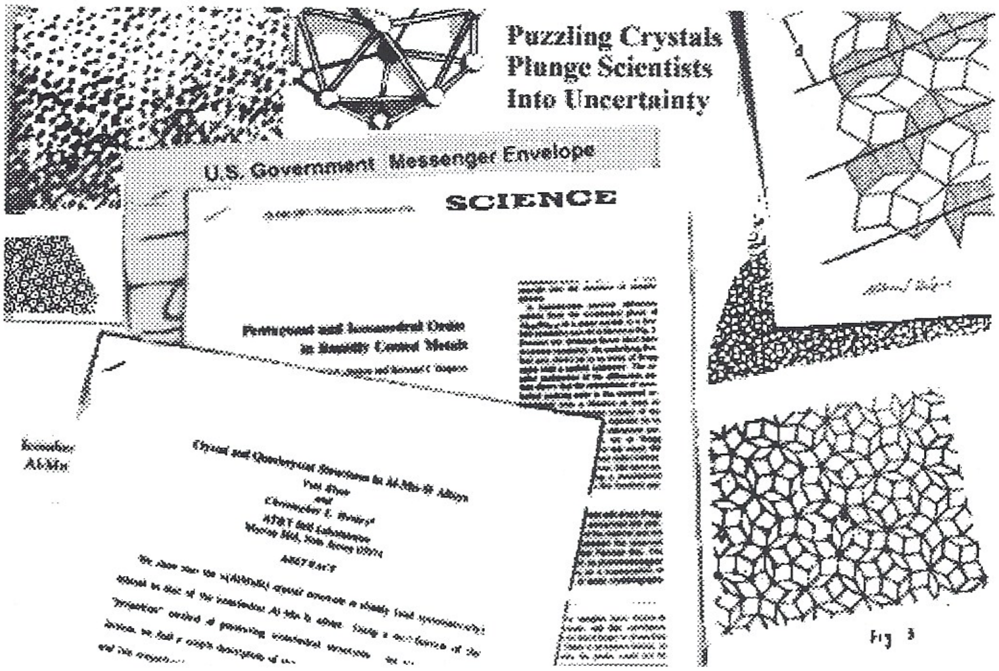
The *Zometool* circle will be pleased to hear from working teachers as though we are colleagues together:

Zometool Teacher's Circle
307 Hungry Hollow Road
Chestnut Ridge, New York
10977

Introduce the *Zometool* carefully into your classrooms: Become acquainted with it in stages.

* *First Stage.* Practice building some geometric models yourself; use them to illustrate your lessons now and then; leave them exhibited in the classroom for the students to see.
* *Second Stage.* Keep a stock of parts available for those students who become fascinated. Allow them to make things in their spare time. Provide some visible places so their models can be seen.
* *Third Stage.* Obtain a larger stock of parts so that whole classes of students can build models simultaneously, perhaps in groups of three.

It is important to think beforehand of how the models are to be displayed. *Zometool* models give a sophisticated, scientific, high-tech appearance to any classroom. Scientists who use the *Zometool* enjoy being photographed with them; and your students will feel proud of their constructions too. They will want them to be seen for a few days and not taken apart right away.


Puzzling Crystals
Plunge Scientists
Into Uncertainty
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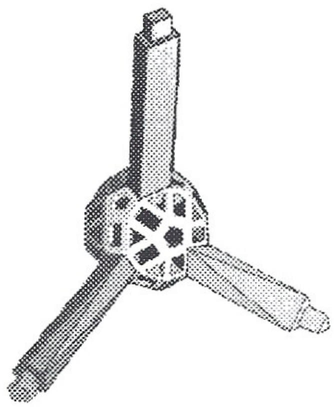
SCIENCE

Section I. All Classrooms

1. Building Tips

Each *Zometool* strut connects to holes of matching shapes. Blue struts fit only into rectangular holes; yellow struts into triangular holes, and red struts into pentagonal holes. This makes it possible to build complex models with ease. There are three different lengths of strut in each color.



Make sure you are putting your strut into a hole of the right shape. See that your strut is turned properly so that you aren't trying to force an edge of a strut into a corner of the hole. Once one of the corners of a strut starts into its hole, the rest of the stub will slip in. The bigger and more complicated your model is the more you need to be certain that the pins on the ends of the strut are pushed all the way, snuggly into the holes.

2. Regular Figures

The representative of all triangles is the equilateral triangle. The representative quadrilateral is the square. The representative pentagon is the regular pentagon, whose angles are all alike and whose sides are congruent. Students need to see these regular figures; they want to have images and not just abstract concepts.

The most difficult figures to show students are the regular icosahedron and the regular dodecahedron; they can, however, be built quickly with the *Zometool*. Before building them practice making some regular polygons. The *Zometool* constructible ones are the polygons with 3 sides (equilateral triangle), 4 sides (square), 5 sides (regular pentagon), 6 sides (regular hexagon), and 10 sides (regular decagon). All of them are made with blue struts only.

Construction 1. *Regular Polygons*

Equilateral triangle: Looking down on the triangle, your *Zometool* connectors will have triangle holes facing up.

[*Inventory Required*: 3 equal sized blue struts, 3 connectors.]

Square: Looking down on the square, you will see rectangular holes.

[*Inventory Required*: 4 equal sized blue struts, 4 connectors.]

Regular Pentagon: The connectors will have pentagons facing up.

[*Inventory Required*: 5 equal sized blue struts, 5 connectors.]

Regular Hexagon: Triangular holes face up.

[*Inventory Required*: 6 equal sized blue struts, 6 connectors.]

Regular Decagon: Pentagon holes face up.

[*Inventory Required*: 10 equal sized blue struts, 10 connectors.]

Diagonals can be inserted into the regular pentagon. In the other polygons there are no *Zometool* constructible diagonals. You will sometimes miss not being able to insert a diagonal into a square. This is the most conspicuous limitation of the *Zometool*. Allowing diagonals of squares would have required much larger connectors.

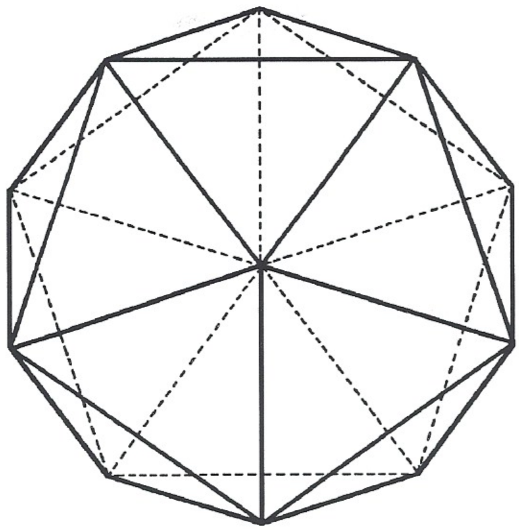
Having made these polygons you are ready to make an icosahedron and a dodecahedron. Let us begin with the icosahedron.

Construction 2. *Regular Icosahedron*

Take one ball and fill all its pentagonal holes with red sticks of the same length (try the middle length at first). Put connectors at the end of each stick – it is actually easier to put a connector on a stick and then put it into the original, central ball. When you have this porcupine (a ball whose pentagonal holes are all filled), connect each of the balls on the outside to its closest neighbors with blue sticks. At some convenient time remove the original inner ball and its red struts.

[*Inventory Required*: 30 middle sized blue struts, 12 connectors.
Temporarily Needed: 12 middle sized red struts, 1 connector.]

When you have done this, you should make a regular icosahedron from scratch, without using the central ball and the porcupine framework.



A regular icosahedron seen corner first.
The edges are all blue struts in the *Zometool* system.

The *regular dodecahedron* should be your next construction project. It can also be made by the porcupine method.

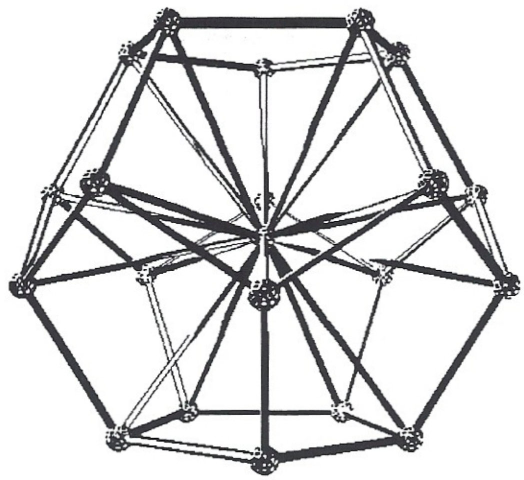
Construction 3 *Regular Dodecahedron*

Take one ball and fill all its triangular holes with yellow sticks of the longest length. Put connectors at the end of each stick – it is actually easier to put a connector on a stick and then put it into the original, central ball. When you have this yellow porcupine (a ball whose triangular holes are all filled), connect each of the balls on the outside to its closest neighbors with *middle sized* blue sticks. At some convenient time remove the original inner ball and its yellow struts.

[*Inventory Required*: 30 middle sized blue struts, 20 connectors.
Temporarily Needed: 20 long yellow struts, 1 connector.]

When you have done this, make a dodecahedron without the help of a porcupine. After finishing the first pentagonal face, stop at look at it carefully for a moment. The two sides of the pentagon are different. One of them will be on the inside of the final dodecahedron and the other will be on the outside.

Appendix A, *The Schläfli Symbol,* gives an account of the Schläfli symbol for the regular polygons and polyhedra. This Schläfli symbol can be taught to school children with good results.



The completed porcupine of Construction 3.

3. Bubbles

Zometool models work well for experiments with soap bubble surfaces. Here are a few hints for dipping your models.

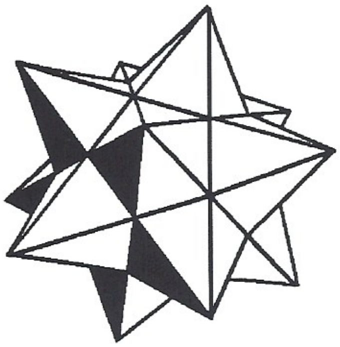
> *Bubble Techniques*
>
> Here are a few hints for dipping your models.
>
> - Fill a deep bucket or bowl with your bubble solution. Use 2/3 cup of dish washing detergent and add enough water to make one gallon.
> - Make sure your container is large and deep enough for your hand.
> - Don't stir up the bubble solution more than necessary. A perfectly smooth surface is best.
> - Dip your models slowly. Pop unwanted bubbles with a dry finger. Move bubbles around without popping with a wet finger.
> - Some models trap bubbles inside. It is interesting to observe that the one model can have more than one kind of interior configuration.

4. Regular Star Polygons

A regular pentagon made with the *Zometool* can be expanded into a pentagonal star. The star's arms are one size larger than the pentagon that forms its core. It your original pentagon is made of short blue sticks, then the arms will be medium blues. If the pentagon is made of medium blues, then the arms will be long blue struts. Appendix B, *The Betsy Ross Fan Club*, gives some drawing suggestions for drawing exercises.

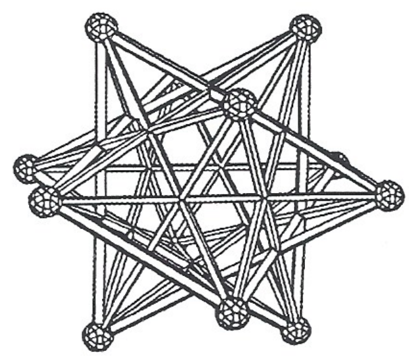
This process can be carried out a dimension higher. Start with a regular dodecahedron and attach mountains on each of its faces. A mountain consists of five blue struts of the next larger size that are joined at a peak.

When all of the mountains are attached the original the core of the completed star. This star has often been used ornamentally; it can be seen in lighting fixtures and Christmas ornaments. In the diagram at the left one of the pentagonal faces has been shaded for emphasis.



> Construction 4. *Stellated Dodecahedron*
>
> Attach mountains to a completed regular dodecahedron (Construction 3). The sides of the mountains are longer than the edges of the central dodecahedron.
>
> [*Inventory Required*: 30 middle sized blue struts, 60 long blue struts, 32 connectors.]

The Schläfli symbol for this figure is {5/2, 5} – see Appendix 1 for the meaning of this symbol.



There is a second regular star polygon that is obtained as a star of an icosahedron.

This star can be built by putting mountains peaks onto the triangular faces of a regular icosahedron. The edges of these mountain peaks extend the sides of the icosahedron just as the mountain peaks of the *stellated dodecahedron,* described immediately above, extend the edges of a regular dodecahedron.

An important new aspect of the *Zometool* arises for the first time when you make a stellated icosahedron: It is the *Fibobbacci rule* of the strut lengths. The arms of the mountain peaks of a stellated icosahedron are *two* sizes longer than the edges of the icosahedron that is its core. If your original icosahedron is a small one made of the shortest struts, then you may use the long struts for the mountain peaks. If, however, your original icosahedron is made up of struts of middle length, then for the mountain peaks of the star, you will need arms that are longer than any in your kit. But that is all right. Longer *Zometool* struts can be made by piecing together shorter ones. The principle of this is the Fibonacci's Rule. Explained in the next section.

> Construction 5. *Stellated Icosahedron*
>
> Attach mountains to a completed regular icosahedron (Construction 2). The sides of the mountains are two lengths longer than the edges of the central icosahedron.
>
> [*Inventory Required*: 30 short blue struts, 60 long blue struts, 32 connectors.]

5. Fibonacci's Rule

The arithmetic text of Fibonacci, from the late Middle Ages in Italy, contained the definition of the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

consisting of numbers having the property that two neighbors in the list add to the next element in the sequence.

Of course, this Fibonacci rule, $x_{n+1} = x_n + x_{n-1}$, could be used beginning with any two initial numbers, 5 and 24 for instance.

5, 24, 29, 53, 82, 135, 217, 352, 569, ...

Let of say that this sequence is a sequence of the *Fibonacci type,* because it satisfies the Fibonacci rule. It is not literally *the* original Fibonacci sequence 1, 1, 2, 3, ..., however.

The sequences of the Fibonacci type that concern us most immediately with the *Zometool* are the three sequences:

short blue strut length,	*middle blue strut length,*	*long blue strut length, ...*
short red strut length,	*middle red strut length,*	*long blue red length, ...*
short yellow strut length,	*middle yellow strut length,*	*long blue yellow length, ...*

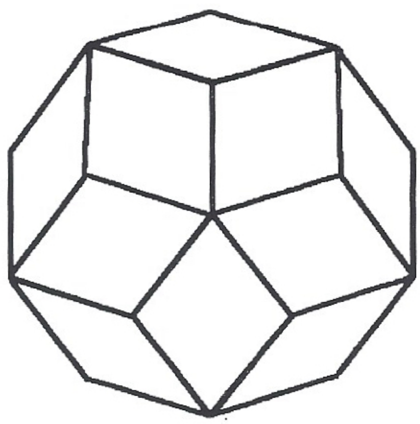
In other words:

short length + middle length = long length

This allows us to make ever longer struts from shorter ones.

the next size strut not included in the kit = middle length + long length

6. Red Struts Only



The rhombic triacontahedron, shown at the right, has two kinds of corners and is made entirely with red lines. The instructions for building this figure, discovered by Johannes Kepler, can be given very easily. Corners having five struts, are connected only to corners having three struts: Corners having three struts are connected only to corners having five struts.

Those are the only instructions that are needed. It is not even necessary for students to see a sample before building the model. Now and then a student will try to make one rhombus that lies in the same plane as its neighbor. That has to be avoided. The rhombii must bend around to make a dome. If you use the long red struts, the final model will be quite large.

Students will see what to do quickly. You will need to prepared to do things with the triacontahedra that they make. The best things to do are to stretch and shrink zones; then when they understand how to do this they can begin to form clusters of tricontahedrons.

Section II. High School Classroom

1. A Quadratic Equation

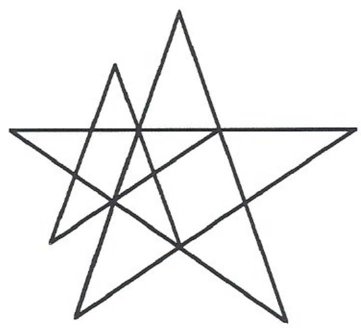
The equation appearing most often in connection with the Zometool geometry is the equation of the golden section ratio.

$$x^2 - x - 1 = 0 \quad \text{or} \quad x^2 = x + 1.$$

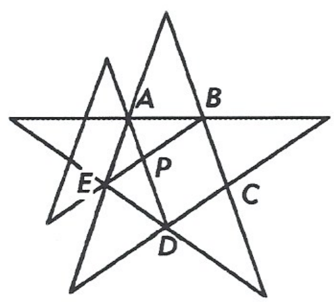
Consider a geometrical progression a, ax, ax^2,... The Zometool struts form a progression of this kind: a is the shortest length, of some fixed color; the next longer length ax is a multiple of it, and so on. But these struts also satisfy the Fibonacci rule that each strut can be replaced by the next two shorter ones.

$$ax^2 = a + ax,$$

In figure below two stars are shown that overlap each other.



The eauation $x^2 = x + 1$ appears here too. First of all label the points



The pentagon $ABCD$ is regular so each of its interior angles, $\angle ABC$, $\angle BCD$, and so on, are equal to 108°. Since triangle ΔAEB is isosceles, it must be that ΔAEB is a 108-36-36 triangle. Likewise, ΔAED is a 108-36-36 triangle. This knowledge allows us to see that ΔAPE is a 108-36-36 triangle too. Triangle ΔAEB is formed by a diagonal across a regular pentagon, thus the similar triangle ΔAPE must be part of a similar, regular pentagon. These thoughts lead us to the small pentagon and its star.

Since the stars are similar to each other, that is they are proportional, we have that:

small neck : *small arm* = *large neck* : *large arm*.

But the diagram also shows us that the neck of the large star is an arm length of the smaller one. Furthermore, the arm of the larger star is an arm and a neck together of the smaller one, so we may get rid of any mention of the large star in the proportion given above. We obtain:

small neck : *small arm* = *small arm* : *small(neck + arm)*.

Now, introducing algebra, let the small neck be taken as a unit length, and let the arm of the small star be given the unknown length x. So our proportion becomes:

$$1 : x = x : 1 + x.$$

Giving us $x^2 - x - 1 = 0$ once again.

2. The Tau Notation, τ

The golden section number is the positive root of $x^2 - x - 1 = 0$. The roots are $\frac{1}{2}(1 \pm \sqrt{5})$. The positive one is $\frac{1}{2}(1 + \sqrt{5})$ which is approximately 1.61803. It may be abbreviated by the Greek letter τ.

3. Why the Artists Sometimes Do Things Differently

The golden section was used by the great artists of the renaissance. But artists often call it ϕ rather than τ, and they use the reciprocal value.

$$\phi = \frac{1}{\tau}$$

Why do they do this? As for the letter ϕ ; it is used in honor of an ancient sculptor; whereas the letter t is merely the first letter of the Greek word for *cut* (a section is a cut).

Why do they use the reciprocal value? Well, suppose that you want to place an important figure in a painting. You do not want it exactly in the middle, that would look stiff, nor at the edge. The renaissance artists believed that the most attractive place to put the central figure was at a golden section multiple to the side. To do this they wanted a value less than one, because the width of the painting could be taken as a unit. This way of thinking has caused artists to emphasize $1/\tau$ rather than τ.

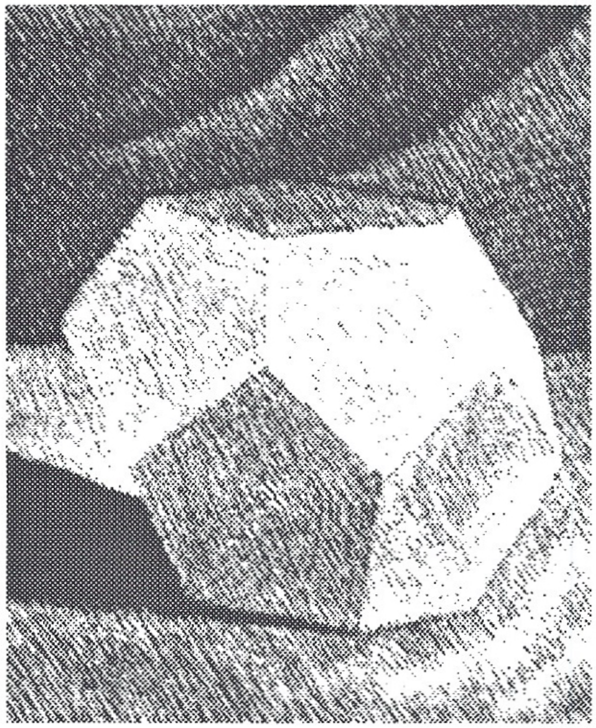
If we get a decimal approximation of $1/\tau$ we obtain:

$$1/\tau = 0.61803...$$

Doing it on a calculator, you find that the numbers following the decimal are exactly those of t. In symbols

$$1/\tau = \tau - 1$$

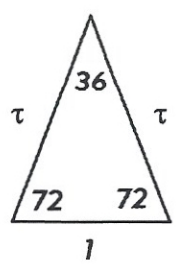
It is not a coincidence at all, because this equation is a thinly disguised version of $x^2 = x + 1$ which is the equation that defined τ in the first place.



Student woodcut from art class (10th grade)

4. Trigonometry with τ

Using the golden section we can immediately gather some trigonometric information. The familiar 72-72-36 triangle forms an arm of the pentagram. Its base is the neck of a star. The sides of this triangle are arms of a pentagram, so its sides must be in the proportion $1:\tau:\tau$. Using the law of cosines, which says that in any ΔABC whose sides are of length a, b, c (side c opposite $\angle C$, and so on) we have that:



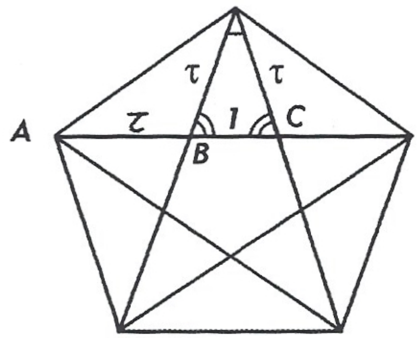
Formula (Law of Cosines) $c^2 = a^2 + b^2 - 2ab \cos C$.

We may apply this generalized theorem of Pythagoras to the 72-72-36 triangle to obtain:

$$\tau^2 = \tau^2 + 1^2 - 2\cdot\tau\cdot 1\cdot \cos 72$$

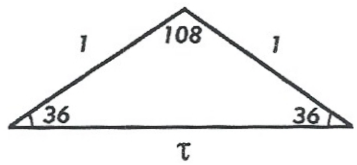
This gives $\cos 72 = \dfrac{1}{2\tau}$.

A second isosceles triangle that appears throughout the Betsy Ross diagrams, the 36-36-108 triangle. The adjacent figure shows a star which is scaled so that its neck has unit length and its arm has length τ. The neck and arm together, segment AC, have length $1 + \tau = \tau^2$.



The triangle ΔABD is a 36-36-108 triangle. This can be seen from the fact that ΔBDC is a 72-72-36 triangle, so that the angle $\angle ABD$ contains 180 − 72 = 108 degrees, this leaves 36 for the two remaining base angles.

Next, turn to ΔACD. By adding the known angles we see that this triangle is a 72-72-36 triangle too. The sides of a 72-72-36 triangle are in the ratio τ:τ:1. Since the short side of ΔACD, DC, has length τ, it must be that the other two sides, AC and AB, have length τ^2. Now we have that the sides of ΔABD are τ, τ, and τ^2: so they are in the ratio $\tau:\tau:\tau^2$. This completes the analysis of the 36-36-108 triangle. The standard triangle of this kind is pictured below.

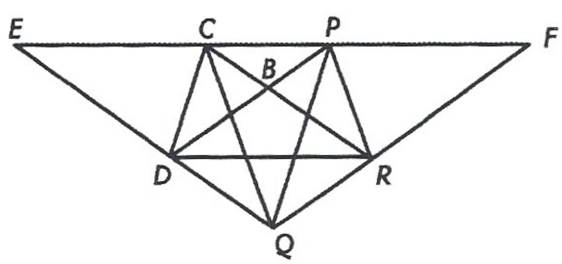


The law of cosines can be applied to this triangle too.

5. Chasing Golden Sections

Students quickly learn to recognize golden section multiples by chasing τ ratios around pentagonal diagrams.

Sample Problem. Given that $AB = 1$, find the length of EF.

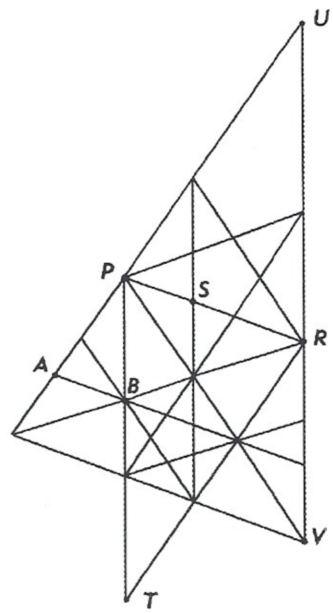


To solve a problem of this kind one can chase ratios around the diagram. With a little experience, shortcuts will soon suggest themselves. First of all, ΔACB is a 72-72-36 triangle, so its sides are in the continued ratio $1:\tau:\tau$, giving us that $CB = CA = \tau$. Now turn to ΔACD which is a 36-36-108 triangle and thus has $CD = \tau^2$. Looking at ΔECD we find that $EC = \tau^3$. There is another 36-36-108 triangle ΔECQ having EC as one of its sides, thus $EQ = \tau^4$. This in turn gives $EF = \tau^5$, finishing the sample problem.

It frequently happens that shortcuts appear when contemplating diagrams of this kind. For example, there is a rhombus *ADQR*; one could pass from the knowledge of one side, across the rhombus to the opposite side, without the need of using a chain of triangles.

A diagram like the one at the right allows for a host of problems that most students find pleasurable, because of the visual interest. This is another topic well suited for classes of mixed ability levels.

1. Given: AB = τ^2. Required: a. SR;
b. TP;
c. UV.



6. Algebraic Simplification

Since the golden section ratio, τ, is given by a quadratic equation, expressions involving τ can be simplified so that τ appears to at most the first power.

This can be used to good effect in classes in which some students need algebra review and others are ready for new iodeas.

A typical problem is:

Simplify: $\tau^5 - \tau^4 + \tau^3 - \tau^2 + \tau - 1$.

After a while the students learn to apply the rule $\tau^n = F_n\tau + F_{n-1}$. So that their calculation would begin $(5\tau + 3) - (3\tau + 2) + (2\tau + 1) - (\tau + 1) + \tau - 1$

Until they realize that this rule allows them to eliminate higher powers immediately, they may want to reduce the exponents a step at a time.

Section III. College Classroom

1. The Cyclotomic Polynomials of Degree Five

The word *cyclotomic* refers to a division of a circle: a division into equal parts. The suffix "tomic" indicates a cutting and its initial letter is the same *t* that appears in Greek as τ, the golden section.

When a unit circle is divided into n equal parts, the cut points that form the division can be represented in the field of complex numbers. Because they are evenly distributed around the unit circle, they will have to be the numbers

$$cos(\frac{2\pi}{n}) + i\, sin(\frac{2\pi}{n})$$

They will be the roots of

$$z^n - 1 = (z - 1)(z^{n-1} + ... + z^2 + z + 1)$$

Many of the cyclotomic polynomials factor further over the field of rationals.

$z^2 - 1 = (z - 1)(z + 1)$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

$z^4 - 1 = (z - 1)(z^3 + z^2 + z + 1) = (z - 1)(z + 1)(z^2 + 1)$

$z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$

$z^6 - 1 = (z - 1)(z^5 + z^4 + z^3 + z^2 + z + 1) = (z - 1)(z + 1)(z^2 + z + 1)\ (z^2 - z + 1)$

$z^7 - 1 = (z - 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$

$z^8 - 1 = (z - 1)(z + 1)(z^2 + 1)\ (z^4 + 1)$

Since the product and quotient of two n^{th} roots of unity is another n^{th} root of unity, the solutions to any one of these equations form a cyclic group. Generators of the group are said to be *primitive roots.* The equation having the primitive n^{th} roots of unity as its roots is the *characteristic equation.*

n	characteristic equation
1	
2	$z + 1$
3	$z^2 + z + 1$
4	$z^2 + 1$
5	$z^4 + z^3 + z^2 + z + 1$
6	$z^2 - z + 1$
7	$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$
8	$z^4 + 1$

When we come to the pentagon, $n = 5$, it will be the first time that the characteristic equation is of degree greater than two.

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Since 5 is prime, any one root of this equation can be obtained as powers of any other: they are all primitive. This equation, possesses a special property: it is a *reciprocal equation*. That is, the coefficients are the same whether read backwards or forwards. This allows us to solve the equation by grouping terms in an appropriate way. The manner in which this is done depends on whether the equation is of even or odd degree. To grasp the method, it is sufficient to consider some examples. The second example is the one that is of interest; it is the cyclotomic polynomial that gives the primitive fifth roots of unity, and therefore the vertices of a regular pentagon inscribed into the unit circle.

Example. Solve: $x^3 + 7x^2 + 7x + 1 = 0$

Solution. $x^3 + 7x^2 + 7x + 1 = x^2(7x + 1) + (7x + 1) = (x^2 + 1)(7x + 1)$. So x = 1/7, $\pm i$.

Example. Solve: $z^4 + z^3 + z^2 + z + 1 = 0$.

Solution. $z^4 + z^3 + z^2 + z + 1 = z^2\left[(z^2 + \frac{1}{z^2}) + (z + \frac{1}{z}) + 1\right]$

Letting $y = z + \frac{1}{z}$, the expression in brackets becomes

$$(y^2 - 2) + y + 1$$

So we can solve equation $(y^2 - 2) + y + 1 = 0$. Or, equivalently $y^2 + y - 1 = 0$. It gives the roots $y = \tau - 1, -\tau$.

This leaves us with the pair of quadratic equations

$$z + \frac{1}{z} = \tau - 1; \qquad z + \frac{1}{z} = -\tau.$$

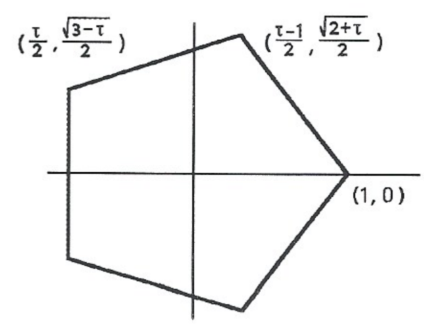
These become: $z^2 + \tau z + 1$ and $z^2 + (1 - \tau)z + 1$.

They provide us with the four imaginary roots:

$$\frac{-\tau \pm i\sqrt{3 - \tau}}{2} = \cos 144 \pm \sin 144$$

and

$$\frac{(\tau - 1) \pm i\sqrt{\tau + 2}}{2} = \cos 72 \pm \sin 72$$



So we are able to calculate the coordinates of the regular pentagon by applying the method solving reciprocal equations to the cyclotomic polynomial of the pentagon. The pentagon that we obtain in this way, Figure 14, can be inscribed in a circle of radius one. Looking at the two points (1, 0) and $(\frac{\tau - 1}{2}, \frac{\sqrt{2 + \tau}}{2})$ we can calculate the length of an edge of this pentagon to be

$$\sqrt{\left(1-\frac{\tau-1}{2}\right)^2+\left(\frac{\sqrt{2+\tau}}{2}\right)^2} = \sqrt{\left(\frac{3-\tau}{2}\right)^2+\frac{\tau+2}{4}} = \sqrt{\frac{12-4\tau}{4}}$$

This is equal to $\sqrt{3-\tau}$. Since $3-\tau$ is slightly less than 1.4, $\sqrt{3-\tau}$ will be somewhat less than 1.2. So the regular pentagon, {5}, that is inscribed in a unit circle has edges of length greater than one. If we want to find the coordinates of a pentagon whose edges are of unit length, it will be necessary to divide these coordinates by $\sqrt{3-\tau}$. Ordinarily, in considering the *Zometool*, the shortest blue struts are assigned a unit length. So we would carry out this scaling by dividing by $\sqrt{3-\tau}$.

Calculations like these involving τ are much easier if you are familiar with the properties of the Fibonacci numbers.

2. Algebraic Integers

The algebraic integers of the number field $\mathbf{Q}(\sqrt{5})$ are, as usual, the roots of a $p(x) \in \mathbf{Z}[x]$ with a leading coefficient of one. It is the same field as $\mathbf{Q}(\tau)$; and the norms also agree.

Since $\frac{m+n\sqrt{5}}{k}$ satisfies $x^2-\frac{2m}{k}x+\frac{m^2-5n^2}{k^2}=0$ we can look for algebraic integers when $k=2$ by checking for the occasions in which m^2-5n^2 is divisible by 4.

m	n	m^2-5n^2	
0	0	0	✓
1	0	1	✖
0	1	−5	✖
2	0	4	✓
1	1	−4	✓
0	2	−20	✓
3	0	9	✖
2	1	−1	✖
1	2	−19	✖

✓= divisible by4
✖=not divisible by 4

m	n	m^2-5n^2	
0	3	−45	✖
4	0	16	✓
3	1	4	✓
2	2	−16	✓
1	3	−44	✓
0	4	−80	✓
5	0	25	✖
4	1	11	✖
3	2	−11	✖

It is interesting to see that these algebraic integers arise frequently in connection with the *Zometool*. A few examples are listed in the following table.

a	b	$a+b\tau$
0	1/2	sin36
−1/2	1/2	sin72
2	0	tanρ
2	−1	τ^{-2}
−1	1	τ^{-1}
1	1	τ^2
2	1	τ^3

The angle ρ in the third line is the angle between adjacent red struts in a *Zometool* node.

3. Some Trigonometry

Definition Let ρ be the angle between adjacent red rays ; and ψ that between adjacent yellow rays.

angle	sine	cos	tan
$\psi/2$	$\frac{1}{\sqrt{3}\tau}$	$\frac{\tau}{\sqrt{3}}$	$\frac{2\tau-3}{\sqrt{3}}$
$\rho/2$	$\sqrt{\frac{1}{\tau+2}}$	$\sqrt{\frac{\tau+1}{\tau+2}}$	$\frac{1}{\tau}$
ψ	$2/3$	$\frac{2\tau-1}{3}$	$2/\sqrt{5}$
ρ	$\frac{2\tau}{\tau+2}$	$\frac{\tau}{\tau+2}$	2

These angles are most easily derived from the knowledge that the angle ρ is found in a $1{:}\tau$ rhombus and the angle ψ in a $1{:}\tau^2$ rhombus. Dividing these rhombi into quarters, we obtain two kinds of right triangles:

$$\pi/2 : \rho/2 : \tfrac{1}{2}(\pi - \rho) \qquad \sqrt{\tau+2} : 1 : \tau$$
$$\pi/2 : \psi/2 : \tfrac{1}{2}(\pi - \psi) \qquad \sqrt{3}\tau : 1 : \tau^2$$

Applying the Theorem of Pythagoras to these triangles reveals the fact that

$$\sqrt{\tau^4 + 1} = \sqrt{3}\tau.$$

Unexpected relationships of this kind frequently appear in calculations of this kind.

In general, $\tau^n + F_{n-2} = (F_n\tau + F_{n-1}) + F_{n-2} = F_n\tau^2.$

Returning to the trigonometry, we can now use these half angle expressions in the table above are known the expressions for the angles themselves may be obtained.

$$\rho = \text{arc}\tan 2$$

This remarkable fact raises the question of how this might have been seen directly. There is a very simple triangle

$$1 : 2 : \sqrt{5} \qquad \rho : \pi/2 - \rho : \pi/2$$

that contains the angle ρ .

Likewise, we obtain the tabulated expressions for ψ :

$$\sin\psi = 2 \sin\psi/2 \cos\psi/2 = 2/3$$

another remarkably simple expression. This means that there is a right triangle

$$2 : \sqrt{5} : 3 \qquad \psi : \pi/2 - \psi : \pi/2$$

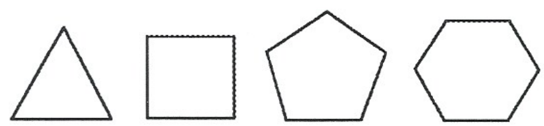
Appendix A. The Schläfli Symbol

It is very helpful to have some kind of notation for regular objects, the geometrical archetypes. H.S.M. Coxeter, the world's foremost authority in geometry, uses the Schläfli symbol. This notation is so simple and clear that it can be put to good use in grade school.

Schläfli

Ludwig Schläfli was a school teacher in the small Swiss city of Thun (you may approximate the Swiss pronunciation of his family name by giving the umlauted *a* the sound of long *a*). Like the subjects many old photographic portraits, he appears very serious. He thoroughly studied the fourth dimension in the middle of the nineteenth century. His work contained no diagrams, was difficult to read, and attracted little notice. It did come to the attention of Jacob Steiner, a famous geometer, who became Schläfli's patron and brought him to Steiner's university.

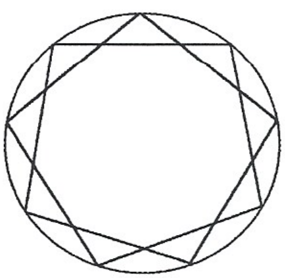
Schläfli's symbol for an equilateral triangle is {*3*}, a square is {*4*}, a regular pentagon {*5*}, and so on.



{3} {4} {5} {6}

For any positive integer *n*, there is a *regular polygon* {*n*} having *n* sides of equal length and *n* congruent angles. These are not symbols for sets; they are simply names for geometrical objects.

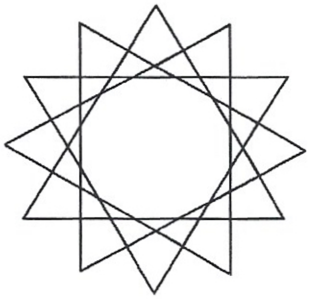
A *regular star polygon* begins, like a regular polygon, with a series of points spaced equally around a circle. To make an ordinary polygon, we join adjacent points. If instead, however, we decide to skip a fixed number of points, then a star results. It will still be regular as long as we always skip ahead by the same number of steps; but it does have edges that cross one another.



The figure at the left shows nine points that have been joined by skipping every other one. This way of skipping is indicated by a *2* in the Schläfli symbol. The number two arises here from several different points of view. We have drawn an edge that passes from a point to the *second* one that follows it. As we trace out the star polygon an observer at the center of the circle has to turn around *twice* while observing the drawing pen. Also, to escape from the center of the star, it is necessary to cross *two* of the straight edges - except at points where they cross of course.

The symbol $\left\{\frac{9}{2}\right\}$, or {*9/2*}, suits this nine pointed star; so we are able to extend the Schläfli notation to include regular star polygons, objects that are treated in sixth grade geometry lessons.

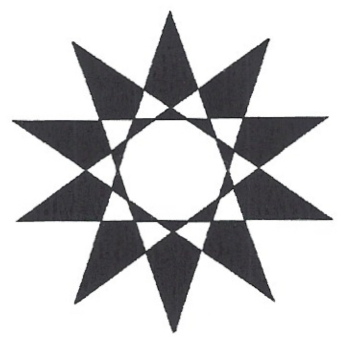
The ordinary nine sided polygon would be {*9/1*}: The *one* means that our edges go from one point to the *next* one, or that the completed figure is enclosed by a boundary that is *one* line thick. Just as we do not write whole numbers as fractions with a unit denominator, we also drop the *1* in this Schläfli symbol; and so the ordinary, regular, nine sided polygon is given the name {*9*}.

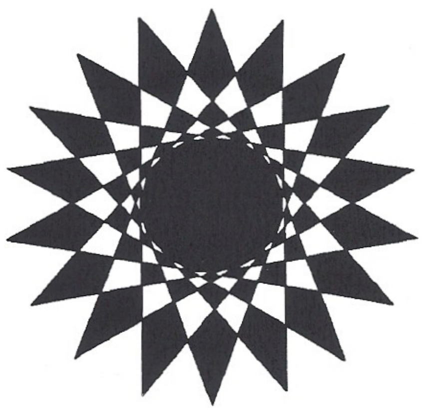


When we draw a {12/4}, shown at the left, we find that the figure breaks down into four separate interpenetrating triangles. A {12/3} will *decompose* into three squares. The hexagon provides the most familiar instance of this decomposition. When we form a {6/3}, the figure decomposes into two interpenetrating equilateral triangles. The resulting symbol is called the *Star of David,* or *Mogan David,* and appears on the flag of Israel.

Students will incorrectly guess that $\{n/k\}$ decomposes into separate figures when k is a divisor of n. The regular decagon provides the simplest counterexample, showing that divisibility is not necessary for decomposition. A {10/4} decomposes into two pentagrams. The numbers 10 and 4 have a divisor, 2, in common. It is the existence of a common divisor, not mere divisibility, that makes stars decompose.

The star {10/4} is shown at the right. It takes a moment to see that it is decomposed into two separate pentagrams. By alternating two colors in adjacent regions of the diagram and then restricting our attention to ever narrower radii, we also see in Figure 5 a white {10/3}, a black {10/2}, and then on the inside, a white {10}.



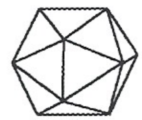


A nested sequence of stars is shown at the left.

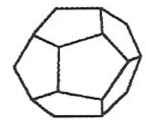
{18/7} > {18/6} > {18/5} > {18/4} > {18/3} > {18/2} > {18}.

The final, inside stages of the sequence are obscured by the size of this diagram. Looking at the outside of the figure, we readily imagine how the edges of the {18/7} could be extended outward to produce an {18/8} that would enlarge the entire figure. This process, of extending edges outward, is known as *stellation*. Beginning with an ordinary regular polygon, we could create stars by stellation, that is by lengthening its edges until they meet.

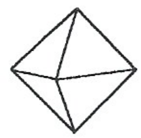
It is a striking fact, known from very ancient times, that there are just five *regular* or *Platonic polyhedra*. A regular polyhedron is a convex body that is bounded by regular polygons and has the property that all of its vertices are alike. The Schläfli symbols for these five polyhedra are {3, 3}, {3, 4}, {3, 5}, {4, 3}, and {5, 3}. These symbols use two numbers in braces instead of a single number, perhaps fractional, that appear in the Schläfli symbols of plane, two dimensional objects.



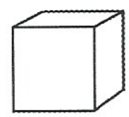
{3,5}



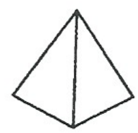
{5,3}



{3,4}

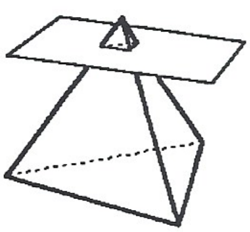


{4,3}



{3,3}

The regular polyhedron {3, 3} is the regular tetrahedron. The first 3 in the Schläfli symbol means that the faces of the figure are all of the type {3}, that is equilateral triangles. The second 3 describes what happens at the corners. If you cut off the corner of a regular tetrahedron, as shown in Figure 8, the section is a triangle. So {3, 3} means that there are triangular faces arranged in a triangular configuration at each vertex.



A cube is {4, 3} because it consists of squares, whose symbol are {4}, meeting three at a corner. The octahedron which is dual to the cube has corners corresponding to cube faces and faces corresponding to cube corners. It has the symbol {3, 4}.

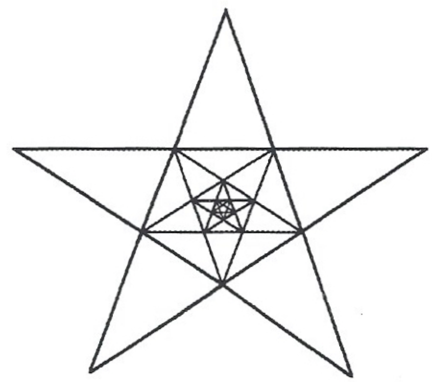
Finally there is the dual pair, {3, 5} and {5, 3}, of the icosahedron and dodecahedron. These two regular polyhedra take longer to understand; they have a mysterious beauty that one does not find in the more familiar, cube-like solids. The icosahedron {3, 5} has triangles that meet five at a vertex; this accounts for the 3 and 5 of its Schläfli symbol.

An eighth grade class will be able to understand more complicated Schläfli symbol, like {5/2, 5}. This figure is composed of pentagrams, {5/2} meeting 5 at a vertex. It is the stellated dodecahedron, construction 4.

Appendix B. The Betsy Ross Fan Club

In the figure at the right, below there is a sequence pentagrams are shown centrally nested within one another. Call it *stars spinning in.*

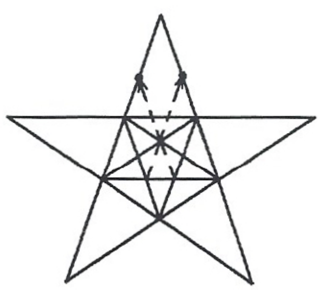
The successively smaller stars alternate point up or point down. Any accurate method of drawing a regular pentagon will do nicely.



The next step is to send *stars down the arms.* It is very easy to introduce inaccuracy at this stage. In general, when we draw a line along a ruler it is much easier to maintain precision when the two points along which our ruler is aligned are widely spaced. When we are aligning along points that are close together the ruler can be jiggled allowing its end to swing while still appearing to maintain its alignment. Our next step requires us to align along near points. Once you understand this step, however, you may very well find other ways to accomplish the same result. These directions work and they are easy to give and to follow. So take a deep breath sharpen your eyesight and let us send some stars down the arms.

The rays, shown in dashes in the figure below, give two new points along the arms of the original, large pentagram. these points form a new pentagon and a new star that goes partially down the arm of the large star.

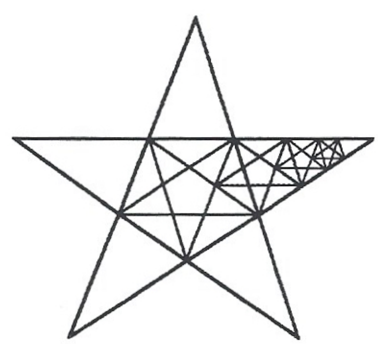
The process can be continued to send stars farther and farther out an arm of our original star.

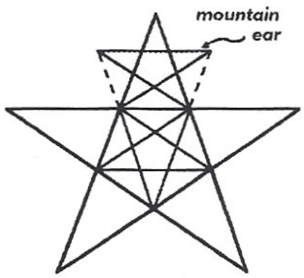


A series of stars going down one of the arms is shown the the figure immediately below.

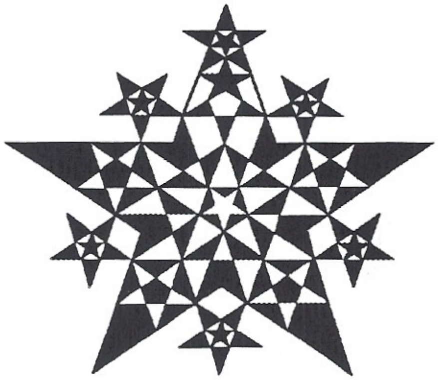
Remember to watch out for inaccuracies. Before drawing a new line check to see if it is parallel to one of the original ones. If not then repeat your steps to find cause of the trouble right away.

To make interesting and attractive designs with interlocking stars it is also desirable to make stars grow larger. This can be done by putting ears onto the arms of a star. Referring to an arm of a star as a *mountain,* we may call them *mountain ears,* as shown below.





The designs that can be made using these techniques: stars spinning in, stars down the arms, and mountain ears are astonishing in their variety. It is possible to color them with only two colors, but three colors is evn more intresting.



When they are patriotically colored red, white, and blue, Betsy Ross would surely approve.

The pupils often take pride in choosing an attractive triple of colors themselves.

Appendix C. Fibonacci Numbers

The Fibonacci numbers are useful for the teacher who needs a topic for use in a class of mixed ability. The stronger students are attracted to the novelty of the number patterns that arise; the weaker ones can be given needed computational exercises in the natural course of the lessons.

The sequence of Fibonacci numbers is given by the recursion $F_{n+1} = F_n + F_{n-1}$ along with the initial conditions $F_0 = 1$ and $F_1 = 1$.

Students working with these numbers need to prepare a list of them.

$F_0 = 0$	$F_{10} = 55$	$F_{20} = 6765$
$F_1 = 1$	$F_{11} = 89$	etc.
$F_2 = 1$	$F_{12} = 144$	
$F_3 = 2$	$F_{13} = 233$	
$F_4 = 3$	$F_{14} = 377$	
$F_5 = 5$	$F_{15} = 610$	
$F_6 = 8$	$F_{16} = 987$	
$F_7 = 13$	$F_{17} = 1597$	
$F_8 = 21$	$F_{18} = 2584$	
$F_9 = 34$	$F_{19} = 4181$	

The teacher needs to have a stock of number patterns that come out of this table and a some classroom method for getting the students to search out the patterns for themselves.

As a first example, let us calculate the Fibonacci sequence backwords, subtracting to find out what F_0 would have to be, then F_{-1} and so on.

Calculating backward according to the rule $F_{n-1} = F_{n+1} - F_n$ we obtain the sequence for negative indices:

$$F_0 = 0 \qquad F_{-1} = 1 \qquad F_{-2} = -1 \qquad F_{-3} = 2 \qquad F_{-4} = -3 \qquad F_{-5} = 5$$

If we write them in a list, we have:

.. . 13, −8, 5, −3, 2, −1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

So the terms with negative indices are like those with positive indices except that every other one picks up a minus sign. That constitutes a pattern. In junior high school we can recognize the pattern and describe it verbally. If we are in high school we can contempate the meaning of the formula

$$F_{-n} = (-1)^{n+1} F_n$$

More advanced high school students, or some college students, can use these formulas a material for practicin proofs by induction.

Historically speaking, the first Fibonacci number pattern was Simpson's rule. Suppose that we were to introduce this rule into the classroom. Simpson's rule is the identity:

$$F_{n+1} F_{n-1} = (-1)^n + (F_n)^2$$

We want the students to discover the rule themselves (remember, this is the historically first, not the logically simplest rule of this kind).

We ask the students to make a table showing the successive values of $F_{n+1}F_{n-1}$. So, at the teacher's suggestion, they fill out a table like the one that follows.

n	F_{n-1}	F_n	F_{n+1}	$F_{n+1}F_{n-1}$
1	0	1	1	0
2	1	1	2	2
3	1	2	3	3
4	2	3	5	10
5	3	5	8	24
6	5	8	13	65
7	8	13	21	168
8	13	21	34	442

Now, we look at this last column and ask: What do these numbers remind us of? The sequence

$$0, 2, 3, 10, 24, 65, 168, 442, \ldots$$

ought to make us think of squares. It is reminds us of 9, 25, 64, 169, 441 = 21^2. It those high school classes that have been trained in arithmetic, some student will make this observation. Classes that have used calculators in grade school might not see this pattern. The teacher has to adapt these lessons to actual conditions. In any event, the students will see this pattern when it is shown to them and they can look for other simpler ones themselves. They can find a way to express the rule, and, if they are old enough to have studied exponents, can write Simpson's rule.

$$F_{n+1}F_{n-1} = (-1)^n + (F_n)^2$$

It is not difficult to prove Simpson's rule by induction.

Here are some rules that can be treated in the same way. The more complicated looking expression on the left side of the identity is put into the tables; the expression on the right is to be obtained by the students themselves.

Rule 2) $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$.

To use this rule in junior high school, avoid the summation notation, and have the students make a table whose entries are F_1, $F_1 + F_3$, $F_1 + F_3 + F_5$ ...

Rule 3) $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$

Rule 4) $\sum_{i=1}^{n} F_i = F_{n+2} - 1$

Rule 5) $(F_{n+1})^2 + (F_{n-1})^2. = F_{2n+1}$

Rule 6) $(F_{n+1})^2 - (F_{n-1})^2 = F_{2n}$

Rule 7) $F_{n+1} F_{p+1} + F_n F_p = F_{n+p+1}$

When $p = 1$, this is the rule, $F_{n+2} = F_{n+1} + F_n$, that defines the Fibonacci sequence. Other specific values of p could be used to generate numerical puzzles for the class.