Zome Geometry

Hands-on Learning with Zome™ Models

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Truly I begin to understand that although logic is an excellent instrument to govern our reasoning, it does not compare with the sharpness of geometry in awakening the mind to discovery.

—Galileo
(said by Simplicio in *Dialogs Concerning Two New Sciences*)

Introduction

Welcome to *Zome Geometry*! We have found the Zome System to be a wonderful tool. With it, we have deepened our understanding of many geometric ideas in two, three, and even four dimensions—especially, but not exclusively, ideas about polyhedra. In this book, we share our favorite activities with students, teachers, and, in fact, any interested reader.

How to Use *Zome Geometry*

This book can be used in several ways.

- *Zome Geometry* activities supplement the secondary curriculum. The prerequisites and the specific curricular connections (from geometry, trigonometry, algebra, and more) are listed in the teacher notes at the beginning of each unit. Some activities preview or introduce the corresponding topics; others are more suited for the review or application of previously studied topics; and many will work both ways.
- *Zome Geometry* can serve as the textbook for a mathematics elective course.
- You can use *Zome Geometry* as a source of projects for math teams, math clubs, or individual students.
- Anyone who wants to explore geometry on his or her own can use *Zome Geometry* as a self-instruction manual. Read the answer only after working on a question, and make use of the teacher notes as you work through the activities.

The time that students invest in laboratory-style activities deepens their understanding and increases their motivation to study geometry. As with laboratory exercises in a science course, the *Zome Geometry* activities provide directions and insightful questions but leave key observations and discoveries to the explorer.

How *Zome Geometry* Is Organized

Each unit focuses on a specific content area and is divided into lesson-size activities. Each activity starts with a Challenge, which students can explore on their own. Preferably, they will do this without access to the rest of the activity, which often answers the Challenge. The Challenge is intended to lay the groundwork by posing a question that will be developed in the guided activity. In some cases, the Challenge is very difficult, and you will need to decide how much time you want to allot for it. Each unit ends with Explorations, which expand or deepen what students have learned in the core of the unit.

If students have built a structure, do not assume that they understand everything about it. While the hands-on work is very helpful in seeing relationships, students may have trouble focusing on concepts while they are building. There are two types of exercises in the core activities—building prompts (1, 2, . . .) and Questions (Q1, Q2, . . .). The Questions give students a chance to reflect, and they increase the likelihood that students will make interesting discoveries while building. Depending on the availability of time and materials, students may do the building exercises in the order given, or they may do a certain amount of building, and then go back to answer Questions. In any case, for students to take full advantage of *Zome Geometry*—and to enable what they learn to transfer to other parts of their mathematical world—it is essential that they come up with written, thoughtful answers to the Questions. Their answers may follow or generate small-group or whole-class discussions.

In general, it is best to keep models intact as long as possible and to keep them within reach of the students. If models need to be stored overnight and shelf space is limited, they can be hung on paper-clip hooks. If you have easy access to a camera, taking pictures of models can be a useful way to document progress and to create records that can be referred to when working on later activities. In some cases, taking photos of half-completed structures yields images that are easier to decipher than pictures of completed projects.

The Explorations that close each unit, and the additional unit of Explorations that closes the book, tend to require more time and materials than the other projects. Their mathematical level is often a bit higher than that of the core part of the activity. Hence, they are well-suited for term papers, extra-credit projects, or challenges for the more ambitious students in a class.

About the Materials

The zomeball is designed with rectangular holes for blue struts, triangular holes for yellow struts, and pentagonal holes for red struts. Green and green-blue struts, which also fit into the pentagonal holes, have been added to the Zome System, so that Zome models of all the Platonic solids can be built. An illustrated list of the strut names is given on page 265. You may want to photocopy this page and display it for reference. Blue, yellow, red, green, and green-blue struts are all available in the special Creator Kit designed for this book and distributed by Key Curriculum Press. If you already have Zome materials, green struts can be purchased separately. In this book, we will refer to both the green and green-blue struts as *green*.

The green struts are challenging to use. Builders should have practice with the blue, yellow, and red struts before they face the challenge of distinguishing between the five different angles that the green struts can fit into a given zomeball pentagonal hole. Follow the instructions in the teacher notes for Unit 3 as you and your students learn to build with the green struts. Most of the activities can be done without the green struts, however. Many involving regular tetrahedra and octahedra made with green struts can be approximated with non-green tetrahedra and octahedra.

The special Creator Kit with green struts includes enough pieces to build all but the most complicated models and the big domes. This kit will be enough for two or three groups to work on many of the activities. However, if you want to build the more complex models or to divide the class into four or more groups, you will need two kits. The index of polyhedra on page 259 indicates the Zome materials required to build each model. Use this list to help you figure out if you have enough materials for a given activity. To conveniently distribute materials at the beginning of class, and to clean up at the end, you may organize your struts and balls in labeled resealable plastic bags. You should have as many bags for each type of strut as you have groups of student builders.

Many of the complex models will take some time to build. If several models are built during an activity, you can conserve Zome materials and time by having each group build a different model and then share the models as students answer the Questions. Also, it is often possible to conserve materials by having different groups of students build the same models in different sizes.

About Frequently Built Polyhedra

Some polyhedra are studied again and again throughout the book. Students will learn to build them quickly. If you embark on an activity out of sequence, however, students may encounter an unfamiliar polyhedron whose building instructions were in a skipped activity. In such cases, consult the index of polyhedra, which lists in bold the activity or activities that contain the building recipe for each polyhedron.

About Numerical Answers

When appropriate, require that students give both calculator-generated numerical answers with reasonable accuracy, such as 1.618, and mathematically exact answers, such as $\frac{1 + \sqrt{5}}{2}$. The former are important if students need to know the magnitude of a number, as for comparison purposes. The latter are important because, in many cases, they facilitate communication and deepen understanding of mathematical relationships.

About Proof

We strongly believe that students should be introduced to formal mathematical proof, but that is not the only thing that needs to happen in math class. Because this book will be used as a supplement to a wide range of math classes, with students at many different ages and levels of mathematical maturity, we have not emphasized formal proof.

Instead, the main purpose of *Zome Geometry* is to introduce students to a beautiful part of geometry, to reinforce their visual sense and spatial intuition, to give them a chance to apply ideas they have learned in other math classes, and to make interesting connections between different areas of mathematics.

We do consistently ask students to reason about the figures they build—a necessary prerequisite to formal proof. Moreover, a few important activities lead students through logically tight arguments about the mathematical properties of the Zome System (see Units 7 and 13) and about three fundamental theorems concerning polyhedra (see Activities 3.3, 24.1, and 24.2).

About the Authors

George W. Hart (http://www.georgehart.com) is an artist and mathematician. Henri Picciotto (http://www.picciotto.org/math-ed) is a mathematics teacher at the Urban School of San Francisco and a curriculum developer specializing in hands-on materials.

A list of updates and corrections for *Zome Geometry* will be available at http://www.georgehart.com.

We hope that you not only learn from this book, but also have a very good time.

UNIT 1

Angles, Polygons, Polyhedra

Students are introduced to the Zome System and its notation. They become familiar with the zomeball and struts and learn which polygons can be built. Then they begin to build in three dimensions and learn how to scale Zome models to produce a similar polyhedron.

Goals

- To become familiar with the Zome System and the notation used to refer to the components
- To discover some of the angles in the Zome System
- To build regular polygons (3, 4, 5, 6, 10 sides)
- To explore the structure of prisms, antiprisms, and pyramids
- To investigate scaling Zome polyhedra

Prerequisites

Students need to know about the angles of a regular *n*-gon.

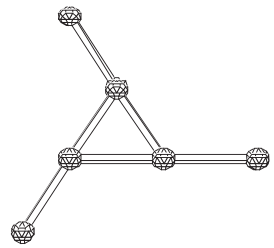
Notes

These models are small and can be made with the red, blue, and yellow struts from any Zome System set. Do not use the green struts until Unit 3.

If your students are not familiar with the angles of regular polygons, use the following discussion to introduce exterior angles.

> A regular *n*-gon has an exterior angle of $360/n$. This can be explained with the following argument. Imagine you are driving a car along the edges of a polygon. At each corner you make a sharp turn, which is the exterior angle at that vertex. For example, if you are driving around a regular hexagon, each turn is 60 degrees. (0 degrees means you don't turn, you just go straight.) When you return to your starting point and turn to face the same way as when you started, you have made one complete revolution, that is, 360 degrees of turns. Since there are *n* equal turns, each turn is $360/n$ degrees.

You may illustrate this argument by making a Zome equilateral triangle, with each side extended as in the figure. A regular pentagon or hexagon would work just as well, but avoid the square as a first example, as its interior and exterior angles are equal.



Triangle with sides extended

Make sure students realize that, at any vertex, the interior and exterior angles add up to 180 degrees.

1.1 Angles and Regular Polygons

Most of the work in this book involves regular polygons, so this introduction is essential. These activities also preview later work on Zome symmetry. Make sure students understand that a regular polygon has equal sides and equal angles and that the angle relationships in the starburst correspond to those in a regular polygon. Suggest that different students build the polygons in different sizes. Advise students to keep the polygons they make, as they will need them later in this unit.

The regular 8-gon requires green struts and is covered in Unit 3. An understanding of why other polygons such as the 7-gon, 9-gon, and 11-gon cannot be constructed with the Zome System follows from the tabulation of all Zome-constructible angles in Unit 13.

1.2 Prisms, Antiprisms, and Pyramids

The activity begins with informal definitions. You may want to mention that the Egyptian pyramids are regular 4-gonal (square) pyramids.

This is the first of many opportunities to count vertices, edges, and faces of polyhedra. As students determine these numbers, encourage them to use logic and think of the structure of a Zome model, rather than count items one by one.

1.3 Zome System Components, Notation, and Scaling

As students become familiar with the zomeball, be sure they notice that for two connected balls, struts inserted in corresponding holes in the two balls will be parallel. Although only certain directions are possible, the set of directions is the same on every ball.

1.1 Angles and Regular Polygons

Challenge

Determine which different regular polygons can be made with the blue, red, and yellow struts of the Zome System. Don't consider different sizes, just different shapes.

1. Take a zomeball and put it on the table so that it is resting on a pentagonal hole. It will also have a pentagonal hole facing straight up, since each hole is opposite another hole of the same shape. Think of the bottom and top pentagonal holes as south and north poles. Stick ten blue struts (any size) into the ten rectangular holes along the equator. They will all be horizontal, defining a flat "starburst" of ten equally spaced rays.

Q1 With a pentagonal hole at the pole, what is the angle between consecutive equatorial struts?

2. Put a triangular hole at the pole, and make a starburst along the new equator.

Q2 With a triangular hole at the pole, what is the angle between consecutive equatorial struts?

3. Put a rectangular hole at the pole, and make a blue starburst along the new equator.

Q3 With a rectangular hole at the pole, what is the angle between consecutive equatorial struts?

Q4 Describe the pattern in the relationship between the number of rays in each starburst and the shape of the hole at the pole.

Q5 Using the angles you found in Questions 1, 2, and 3, you should be able to make regular n-gons for five different values of n. What are the values of n?

4. Build any kinds of regular polygons you didn't already build during the Challenge. Look for polygons with different shapes, angles, and numbers of sides. You don't have to build different sizes.

Q6 Place the regular polygons on the table, and make a chart relating the shape of the hole at the zomeball pole with the corresponding n.

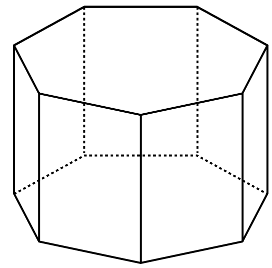
A more advanced study of zomeball angles shows that there are no other regular polygons constructible with the red, yellow, and blue struts. The regular 8-gon will be constructed using the green-blue struts.

1.2 Prisms, Antiprisms, and Pyramids

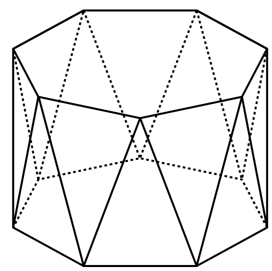
Challenge

Determine which right prisms, which antiprisms, and which pyramids can be built with the Zome System.

A three-dimensional analog to the polygon is a prism. A *right prism* is shaped something like a drum, but with an *n*-gon (instead of a circle) for top and bottom and rectangles around the sides. An *antiprism* is a fancier kind of drum-shaped polyhedron. It also has an *n*-gon as top and bottom, but they are rotated with respect to each other so that the vertices of the top one are between the vertices of the bottom one. The sides of an antiprism are triangles instead of rectangles.



7-gonal prism



7-gonal antiprism

An *n*-gonal pyramid has an *n*-gon for a base and *n* triangles for sides. The vertex where all the triangles meet is called the *apex*.

1. Make a regular pentagon for a base (use any size of blue struts). Then place a red strut (any size, but all five the same) into the north pole of each of the five zomeballs. Top each red strut off with another zomeball, and connect them to make a second regular pentagon. Your pentagonal prism is a three-dimensional solid bounded by two pentagons and five rectangles. The shape of the rectangle depends on which size struts you chose.
2. Make a prism using a triangle, a square, a hexagon, or a decagon as base. (The sides will be rectangles, and, in the case of the square base, they can be squares.) Make a prism different from your neighbors'.

Q1 Build a square prism with square sides. What is another name for this polyhedron?

A *vertex* is a corner of a polygon or polyhedron where the edges meet. It is represented by a zomeball. The plural of *vertex* is *vertices.*

Q2 For an *n*-gonal prism, write formulas that involve *n:* for the number of vertices, for the number of edges, and for the number of faces.

3. Make a pentagonal antiprism: Connect the top and bottom pentagons by a zigzag of edges, making ten equilateral triangles, half pointing up and half pointing down. (If you have trouble doing that, turn your base pentagon upside down.)

There are five different shapes of pentagonal antiprisms that you can make with the lengths and angles in the Zome System. They differ according to how far you raise the top pentagon and which size strut you use to connect the two pentagons.

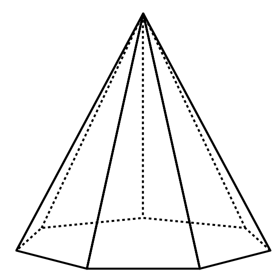
4. With your neighbors, make the other four Zome pentagonal antiprisms. Look for different shapes and angles, not different sizes. The zigzag might be red, yellow, or blue. (Hint: One is very short. Remember that turning over your first pentagon may help.)

5. With your neighbors, build five different shapes of triangular antiprisms. The zigzag might be red, yellow, or blue.

Q3 For an *n*-gonal antiprism, write formulas that involve *n:* for the number of vertices, for the number of edges, and for the number of faces.

6. Make a Zome square pyramid.

Q4 How many different Zome pyramids can you make on an equilateral triangle base? (Hints: Try a base made with the medium-size blue strut. Once you have exhausted the possibilities with one side of the base up, turn the base over.)



7-gonal pyramid

Q5 How many different Zome pyramids can you make on a regular 5-gon base?

Q6 For an *n*-gonal pyramid, write formulas that involve *n:* for the number of vertices, for the number of edges, and for the number of faces.

1.3 Zome System Components, Notation, and Scaling

Challenge

Make antiprisms in four different sizes, but with the same angles.

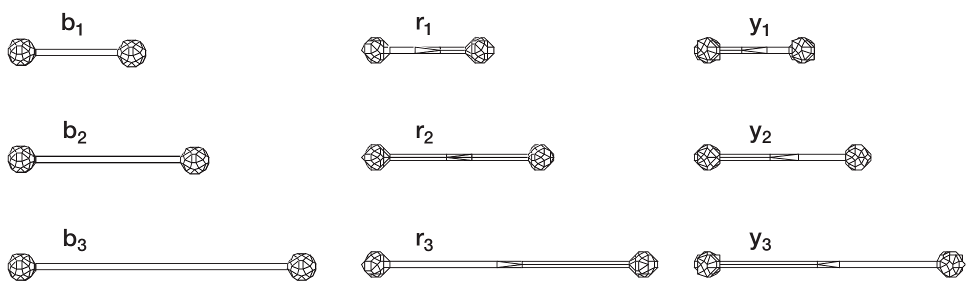
In making prisms, you relied on an important property that is designed into the Zome System: Whenever balls are connected, they have the same orientation. As a result, you can always construct a line parallel to any given strut from any other connected ball.

Q1 Hold up a zomeball and look into a rectangular hole, through the center of the ball, and out the opposite rectangular hole. Notice that the long sides of these two rectangles are parallel. Do the same with a triangular hole and a pentagonal hole. What do you notice?

Q2 Hold any blue strut vertically and examine it. Notice that, disregarding the ends that fit into the zomeball, it can be seen as a tall prism with a rectangular base. Do the same with any yellow strut and any red strut, and notice that each is composed of the connecting ends and three distinct structures in a stack. What are the three distinct structures of these struts?

Q3 Describe what the twists in the red and yellow struts have to do with the zomeball.

The struts are named as shown in the figure, with **b** for blue, **r** for red, and **y** for yellow. For each color, there are three sizes, which are numbered 1, 2, and 3.



Zome struts

Scaling a polyhedron can be done by adding 1 to each size of strut. For example, if you made an antiprism using b_2s for the pentagons and r_1s for the zigzag, the next larger size uses b_3s and r_2s in the corresponding places. When scaling a figure, be sure that angle measures are unchanged!

1. Pick one of the five pentagonal antiprisms you made, and make one similar to it, but scaled up or down one size.

Another approach to scaling is to double (or triple, and so on) the number of struts on each edge. For example, if you had edges of length b_2, connect two b_2 struts with a ball, and use length $2b_2$ for the corresponding edges in the scaled polyhedron.

Q4 Visualize the shape of a cross section of a pentagonal antiprism halfway up between the base and the top. What kind of polygon is it?

2. Build a model that explicitly shows this cross section. Pick any one of the five pentagonal antiprisms and build it in double scale. The balls at the halfway points of the zigzag struts are just where you need them to make the cross section.

3. Make a triangular antiprism in double scale to show its cross section halfway up.

Q5 In general, what polygon do you see if you slice an antiprism with a cut parallel to and halfway between its *n*-gons?

Q6 Explain why you cannot use the Zome System to make an antiprism with a 6-gon or 10-gon base. You may assume that 12-gons and 20-gons are not Zome-constructible.

Explorations 1

A. Skew Polygons A *regular skew polygon* is a polygon with equal sides and equal angles that does not lie in a plane. Instead, its vertices lie on an imaginary cylinder, in two parallel planes. Make a cube and let it hang down from a vertex; notice that around its "equator" is a regular skew hexagon. The zigzag of an antiprism is a regular skew polygon. How many different kinds of regular skew polygons can you build with the Zome System?

B. Nonright Prisms The prisms you made are *right prisms* because the edges connecting the two *n*-gons are at a right angle to the plane of the base. Find some Zome nonright prisms.

C. Another Cross Section What is the cross section of an antiprism one third of the way from one base to another?

D. Rhombic Pyramids A *rhombus* is a planar 4-gon with equal sides, but the angles do not need to be equal. The plural of *rhombus* is *rhombi.* Find four different kinds of blue rhombi, meaning that they have different angles from each other. A square is one of the four, since it is a special kind of rhombus. You should be able to determine the angle in each of your four blue rhombi. There are also one kind of red rhombus and two kinds of yellow rhombi. Build them.
A rhombic prism is easy to build; as more of a challenge, can you make a pyramid on each of your rhombi? (The apex need not be directly over the center of the rhombus.)

E. Concave Antiprisms The prisms and antiprisms above are all convex—they have no indentations. You can also make a symmetric concave antiprism. Start by making a 5-gonal antiprism with b_1 pentagons and b_1 slanting edges. (Its ten sides are equilateral.) Think of the zigzag as a cycle of ten struts numbered 1, 2, 3, . . . , 10, and remove the odd-numbered ones. That creates five openings, each like a rhombus, but not planar. Put in a b_2 as the long diagonal of each "rhombus." Describe the result. How is it like and unlike the other pentagonal antiprisms?

UNIT 2

Icosahedron and Dodecahedron

Students are introduced to the icosahedron and dodecahedron and use these regular polyhedra to explore the effects of scaling.

Goals

- To learn about the structure of the icosahedron and dodecahedron
- To understand proportionality of similar polyhedra
- To become familiar with other polyhedra related to the icosahedron and dodecahedron

Prerequisites

The first part of this unit has no prerequisites. Familiarity with the concept of similarity and scaling is necessary for the second half, although it is also possible to do the unit while first studying similarity and scaling.

Notes

The word *polyhedron* was used in Unit 1 without a definition. There are many ways to define it. Since the Zome polyhedra are frameworks of edges and vertices, this book emphasizes the vertices and edges as determining the polyhedron. However, the word *polyhedron* suggests a definition that emphasizes the faces such as "a solid bounded by plane polygons." The names of specific polyhedra also emphasize the face. You can introduce the words *icosahedron* and *dodecahedron* as coming from the Greek *icosa* meaning 20, *dodeca* meaning 12, and *hedron* referring to the faces.

The polyhedral constructions in this unit involve building some scaffolding or intermediate structure as an aid to constructing the final form. In the end, the scaffolding is removed and just the intended polyhedron remains. If students see the pattern that develops after making some of the scaffolding, it is not necessary to make all the scaffolding; they can just continue the pattern to make the final form. If students get confused, they can go back to including the scaffolding.

Several of the polyhedra in this unit take considerable time and material. Student groups should make the first icosahedron and dodecahedron in different sizes. Then the groups' models can be saved and combined to build the larger structures in Activity 2.2.

2.1 Building and Counting

Instead of using the scaffolding strategy, you may suggest that students use what they learned in Unit 1 about regular polygons in order to build the faces.

The relationship between the numbers of faces, edges, and vertices will be discussed in Unit 6, and the answer to Question 6 will be explored further in Unit 9.

2.2 Scaling

This activity reviews the basic concepts of similarity and scaling, and previews the additive relationship between the Zome struts, which will be pursued in Unit 7.

Remind students that in making similar polyhedra, all lengths scale by the same amount. These include

- the distance between opposite faces
- the distance between opposite vertices
- the distance between opposite edges

Students can see all three of these scaling relationships at once with the construction of concentric dodecahedra.

In a $\mathbf{b}_1$ icosahedron, the distance between opposite vertices can be built with two $\mathbf{r}_1$s, and the distance between opposite edges can be built with a $\mathbf{b}_2$. However, there is no Zome length for the distance between the opposite faces of this icosahedron. You may discuss scaling by having students insert the appropriate struts in different-size icosahedra to show these lengths and their relationships.

The challenge is difficult. It is answered in Question 3. The scaled dodecahedra on the color insert display the answer for a similar question about the dodecahedra.

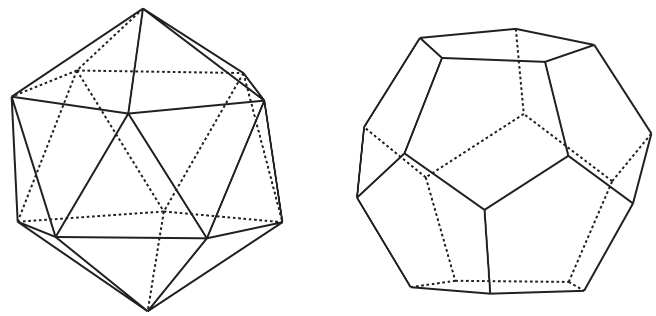
Explorations 2

After making any of these polyhedra, students should count the number of faces, edges, and vertices, and record the numbers for discussion in a future lesson. You may want to keep a class list of all the polyhedra students build, recording the name or description of the polyhedron, *F, E, V,* and perhaps the name of the student builder.

2.1 Building and Counting

Challenge

The icosahedron consists of 20 equilateral triangles. The dodecahedron consists of 12 regular pentagons. Build them both.



Icosahedron and dodecahedron

1. If you did not succeed in building it on your own, here is a method to build the icosahedron using red struts as scaffolding. Make a three-dimensional starburst by putting red struts (all the same size) in all the pentagonal holes of one zomeball. Put another zomeball on the end of each red strut and connect them with blue struts. The blue is the icosahedron, so remove the red struts and the central ball.

   Q1 For the icosahedron, give the number of faces, edges, and vertices.

   Q2 How many edges does each face have? How many edges meet at each vertex?

   Q3 Notice how the icosahedron can be seen as an antiprism with two pyramids glued on. In how many different ways can you find a pentagonal antiprism in an icosahedron?

2. If you did not succeed in building it on your own, here is a method to build the dodecahedron using yellow struts as scaffolding. Make a three-dimensional starburst by putting yellow struts (all medium or all large) in all the triangular holes of one zomeball. Put another zomeball on the end of each yellow strut and connect them with blue struts. The blue is the dodecahedron, so remove the yellow struts and the central ball.

   Q4 For the dodecahedron, give the number of faces, edges, and vertices.

   Q5 How many edges does each face have? How many edges meet at each vertex?

   Q6 Compare the numbers of faces, edges, vertices, edges on each face, and edges that meet at each vertex in a dodecahedron to those in an icosahedron.

2.2 Scaling

Challenge

Which is taller: a b_3 icosahedron or a b_1 icosahedron stacked on top of a b_2 icosahedron?

1. Make a Zome model of two regular pentagons, such that pentagon *B*'s edge is twice as long as pentagon *A*'s. Include a diagonal in pentagon *A*.

   Q1 Predict the size of the diagonal in pentagon *B*.

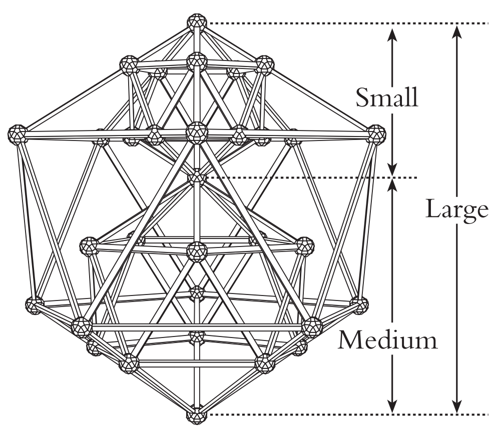
2. Check your prediction by building the diagonal.

   Q2 What is the scaling factor, or ratio of similarity, between pentagon *B* and pentagon *A*?

3. Combine two (or even three) dodecahedra with different edge lengths, smaller inside larger, with a common center. Use some radial yellow struts, as in the starburst, to connect them. Adding a ball at the very center and just two radial struts to the inner dodecahedron, construct and point out similar triangles.

If your eyes were at the center, your views of the dodecahedra would exactly overlap. The scaling factor (ratio of similarity) is b_3/b_2 or b_3/b_1 or b_2/b_1, depending on which pair of dodecahedra you are discussing.

4. Build small and medium icosahedra that touch at only one point, side-by-side inside a large icosahedron. (Hint: The b_1 and b_2 icosahedra have one vertex in common and are built on diametrically opposite sides of that vertex. At the vertices farthest from the common vertex, you need to extend the b_1 and b_2 triangles into b_3 triangles. To do this, notice that you can make a b_3 edge by connecting a b_1 and a b_2 in a straight line: $b_1 + b_2 = b_3$.)



Three icosahedra

   Q3 Hold your model three ways, standing it on a vertex, a face, and an edge. Describe how the sum of the small and medium add to the large for each distance between faces, between edges, and between vertices.

5. If you have time, you can make an analogous compound (small and medium sharing a vertex inside large) for the cube and the dodecahedron.

Explorations 2

Each of the following constructions leads you to another polyhedron related to the icosahedron and dodecahedron. After making any of these polyhedra, count the number of faces, edges, and vertices, and record the numbers for discussion in a future unit.

A. **Nonregular Icosahedron** Instead of using 30 blue struts, make an icosahedron using 10 blue, 10 red, and 10 yellow. This is a nonregular icosahedron, made of 20 triangles, but not equilateral triangles. It has the same topology as the regular icosahedron (the same number of edges, faces, and vertices, and they are connected in the same way) but different geometry (lengths and angles). There are several solutions.

B. **Nonregular Dodecahedron** Now make a nonregular dodecahedron from 10 blue, 10 red, and 10 yellow struts, topologically the same as a regular dodecahedron, but geometrically different. Again, there are several solutions.

C. **Elevated Dodecahedron** Take a regular dodecahedron and erect a blue pentagonal pyramid on each face. You need to use the next larger size strut (such as b_2 struts on a b_1 pentagon or b_3 struts on a b_2 pentagon), so the original dodecahedron must be small or medium, not large. You will be adding a new ball outside the middle of each face. When done, you have a nonconvex polyhedron consisting of 60 isosceles triangles.

D. **Concave Equilateral Deltahedron** Build a dodecahedron and erect a blue pentagonal pyramid on the inside of each face. You will be adding a new ball slightly inside the middle of each face. When done, you have a polyhedron consisting of 60 equilateral triangles. *Deltahedron* means made of triangles, not necessarily equilateral.

E. **Rhombic Triacontahedron 1** Build an icosahedron and erect a shallow red triangular pyramid on the outside of each face. You need to use the next smaller size strut (such as r_1 struts on a b_2 triangle), so the original icosahedron must be medium or large, not small. You will be adding a new ball slightly outside the middle of each face. Then remove the icosahedron. What is left will have rhombic faces.

F. **Rhombic Triacontahedron 2** Build a dodecahedron and erect a shallow red pentagonal pyramid on the outside of each face. You will be adding a new ball slightly outside the middle of each face. Then remove the dodecahedron. This gives the same result as the method in E!

UNIT 3

The Platonic Solids

With the green struts, students can build the two remaining Platonic solids, the tetrahedron and the octahedron. (Without the green struts, they can build approximations of these two regular polyhedra.) Then students prove there can be only five completely regular polyhedra and explore shapes made from truncating the Platonic solids.

Goals

- To become familiar with the green struts
- To learn more about the Platonic solids
- To see some interrelationships between the Platonic solids
- To prove that there are only five Platonic solids
- To begin to learn about some Archimedean solids

Prerequisites

Units 1 and 2

Notes

As a preparation for this unit, students can make paper models of all five regular polyhedra (sometimes called the *Platonic solids*). Since they have built the dodecahedron and the icosahedron in Unit 2 and the cube in Unit 1, they should make paper models of at least the tetrahedron and the octahedron by cutting out equilateral triangles from heavy paper and taping them together along the edges. They can also become familiar with these five regular polyhedra by drawing them.

3.1 Green Polygons

The green struts (*green* refers to both green and green-blue struts) work differently from the other types of struts. They do not extend straight out of a hole specific to them. They fit into the holes designed for the red struts (the pentagonal openings) and tilt at an angle. Make sure your students understand that, like the other struts, the green struts model straight line segments. The purpose of the tilt at the end is to make it possible to aim the struts in a direction that is unavailable as a hole in the zomeball. Carefully point out that when two green struts are extensions of each other on either side of a zomeball, they are correctly aligned, in spite of the little zigzag that can be seen near the ball.

Teacher Notes

There are five ways to rotate a green strut in any of the 12 holes designed for red struts, and therefore a green strut can point in any of 60 directions. Because there are so many possibilities, and because only one of the five possibilities can be used in a model at one time, green struts can be tricky. Point out that when a green strut is inserted into a zomeball, it has a *pointy* side that lines up with a vertex of the pentagonal hole and a *flat* side that lines up with a side of the pentagonal hole. A fine point that students may appreciate after making the regular octahedron is that the sides of the strut (if you think of it as a tall rhombic prism) are in the planes of the octahedron's faces. Knowing this helps students rotate the strut properly when making 60-degree angles without scaffolding.

In this unit, blue struts are used as scaffolding whenever a green strut is needed. This works because every green strut is the hypotenuse of an isosceles right triangle with blue legs.

Most of this book does not require green struts. The exercises that do use green struts say so explicitly. Alternate versions of those exercises are often given. The first activity is a necessary introduction to the green struts.

3.2 The Regular Tetrahedron and Octahedron

Once students complete this activity, they will be able to build Zome models of all five Platonic solids. These models are pictured on the color insert.

3.3 Only Five Platonic Solids: A Proof

You may want to share with your students some history of the Platonic solids. The ancient Greeks discovered that there are exactly five polyhedra with the following properties:

- Each face is a regular polygon.
- Each face is identical.
- Each vertex is identical. This is a shorthand way of saying that at each vertex, the same kind and number of faces meet in the same way.

Plato wrote about the regular polyhedra in his book *Timaeus,* from which they received the name *Platonic solids.* In *The Elements,* Euclid wrote about them and proved that there are only five.

Several students can collaborate to build the models that satisfy the Challenge.

3.4 Truncation

There are many interrelationships between polyhedra. In this unit, students encounter one: the *truncation* of one polyhedron to form another one. Truncation can be thought of as slicing off the corner of the solid. The figure on the activity page shows the truncating of one corner of a cube to create a new triangular face and change the three squares that met at that face into irregular pentagons. You may want to explain to students that if *X* is a polyhedron, then a *truncated X* is a new polyhedron formed by truncating all the corners of *X*. Ask them to visualize a truncated cube. It has eight triangular faces (corresponding to the eight vertices of the cube) and six 8-gon faces because six 4-gons change into 8-gons when four vertices are truncated.

A related process is called *truncating to the edge midpoints.* This involves taking a deeper cut, down to the midpoints of the edges, so that the new faces meet at vertices.

The truncated Platonic solids are examples of Archimedean solids; they will be seen again throughout this book, especially in Unit 12.

3.1 Green Polygons

Challenge

Use eight identical green struts to build an equilateral square pyramid.

1. To make an isosceles right triangle, make a right angle of two b_1s and three zomeballs. A g_1 fits as its hypotenuse.

The shorter green strut is the length of a b_1 and is called a gb_1 strut. (Because green and green-blue struts have the same shape, the word *green* refers to either.)

2. To make a green square, make a $2b_1$ square and connect the four edge midpoints with g_1s. Notice that green squares are rotated 45 degrees from the blue squares.
3. To make a green and blue starburst, place eight struts into a zomeball so that they lie in a plane, separated by 45 degrees.
4. To make a green equilateral triangle, put three b_1s into one zomeball so that they are mutually perpendicular, that is, like one corner of a cube. Make the three right angles into three isosceles right triangles with g_1s. Removing the blue leaves just a green triangle.
5. To make a green regular hexagon, first make a $2b_1$ cube. Find and construct a green slice of it that makes a hexagon. (Hint: It may help to rest the cube on a vertex, with a long diagonal vertical.)

   Q1 Where do you add green struts so that, when you eliminate the blue, what remains is a regular hexagon?
6. To make a green 60-degree starburst, place six green struts into a zomeball so that they lie in a plane, separated by 60 degrees. Notice that three of the red holes are slightly above the plane and three are slightly below it. The green struts tilt down when starting in a hole too high and tilt up when starting in a hole too low.
7. To make an irregular octagon, make a $3b_1$ square and cut off each corner with a g_1 diagonal. Then remove the corner blue struts so that just an 8-gon remains. Cutting off the corners of a polygon is called *truncation.*
8. Noting the orientations of the blue and green struts in your irregular 8-gon, make a copy of it, but with gb_1s instead of g_1s.
9. Construct another green square, green equilateral triangle, green regular hexagon, and regular octagon, but this time do not use blue struts as scaffolding. For the octagon to be regular, gb_1 struts must be used, but the other polygons can be made with any size of strut.

   Q2 All of these green polygons can be raised into prisms. However, with the Zome lengths, only two can be made into a prism with square faces on the sides. Which ones?

3.2 The Regular Tetrahedron and Octahedron

Challenge

Build a regular tetrahedron and a regular octahedron using green struts. All faces of both solids are equilateral triangles.

1. Use green struts to build a regular tetrahedron. First make a b_1 square and insert a g_1 as its diagonal. Then extend this into a blue cube with one green diagonal in each of the cube's faces, forming a green tetrahedron. Remove the cube edges so that only the tetrahedron remains.

   If you do not have green struts, a triangular pyramid with b_1 base and r_1 sloping edges approximates a regular tetrahedron. (The apex angle of the red and blue isosceles triangles is about 63 degrees, close to 60.)

2. Use green struts to build a regular octahedron. First, make a perpendicular blue starburst—that is, six b_1 struts in one ball—forming three mutually perpendicular lines (like x, y, z axes). Add six balls at the ends. Connect the 12 right angles with 12 g_1 diagonals to make a green octahedron.

   If you do not have green struts, a triangular antiprism with two equilateral b_1 triangle bases and an r_1 zigzag approximates the regular octahedron.

   Q1 The regular octahedron is a special equilateral case of what type of polyhedron?

   Q2 The edges of the regular octahedron can be viewed as forming what polygons besides triangles and in what arrangement?

3. The tetrahedron in the cube uses only half the vertices of the cube, four of the eight. If you use the remaining vertices, there is a second tetrahedron in the same cube. Using green struts in a double-scale cube, make a model of two tetrahedra in a cube. First, make a green *X* inside a $2b_1$ square. Expand this into a $2b_1$ cube with a green *X* in each face. Notice how the two tetrahedra pass through each other, crossing at their edge midpoints.

4. To make a compound of two tetrahedra, remove the blue cube edges to leave just two intersecting tetrahedra. This is usually called the *stella octangula,* Latin for "eight-pointed star." Each point of each tetrahedron pokes through a face of the other tetrahedron. (To have your stella octangula stand up on a point, place three r_1s into one vertex, forming a tripod stand.)

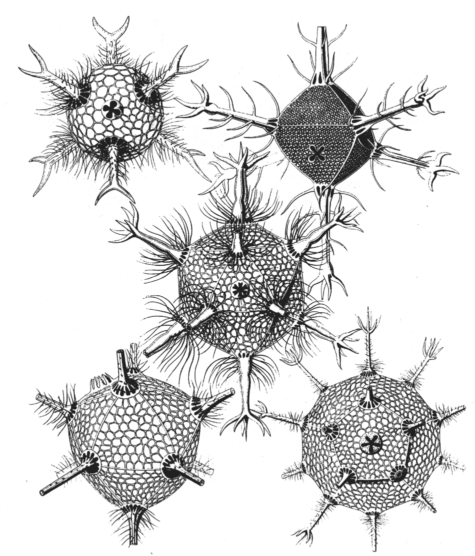
   Q3 Visualize the shape that is the intersection of these two tetrahedra. What shape is it?

5. Connect the crossing points of your stella octangula with 12 green struts to construct the intersection of the two tetrahedra. Notice that your model can now be seen as an octahedron surrounded by eight small tetrahedra.
6. Remove four small tetrahedra to leave one large tetrahedron. This shows how a tetrahedron can be dissected into an octahedron and four smaller tetrahedra.

(Here is an alternate approach to the stella octangula, without green struts. A b_x equilateral triangle with three r_x edges added makes a triangular pyramid that is almost a regular tetrahedron. You can choose x to be 1, 2, or 3. Make this in double-scale ($2b_x$ base and $2r_x$ slanting edges) and connect all the edge midpoints. The central region is almost a regular octahedron, and it is surrounded by four small nearly regular tetrahedra. Add four more blue and red tetrahedra to the octahedron to make a nearly regular stella octangula. You can see the almost-cube it sits in, but you cannot build its edges, as the Zome System does not include directions that would go straight from one "cube" vertex to another in this structure.)

7. The regular tetrahedron can be sliced into two equal parts to reveal a square cross section. Make a model of this.

Connection



Microscopic radiolarian skeletons sometimes take the form of Platonic solids. Drawing by nineteenth-century naturalist Ernst Haeckel.

Source: Hermann Weyl, Symmetry. *Copyright © 1946, renewed 1952. Reprinted by permission of Princeton University Press.*

3.3 Only Five Platonic Solids: A Proof

Challenge

Build polyhedra that are almost Platonic, in that each one violates only one of the three required properties. (Each face is a regular polygon; each face is identical; and at each vertex, the same number of faces meets.)

Q1 Make and fill out a table like this:

Polyhedron	Sides on each face	Faces at each vertex	Faces	Vertices	Edges
tetrahedron					
octahedron					
cube	4	3			
icosahedron					
dodecahedron					

Q2 Looking at the table, notice how the icosahedron and dodecahedron are a pair in certain respects. Name the other two that are a pair in an analogous way, and describe the partner of the remaining one.

To prove that there are only these five possibilities, think about the sum of the angles at each vertex and examine the first two columns of the table. The notation {4, 3} designates a polyhedron whose faces have 4 sides with 3 faces meeting at each vertex; {4, 3} refers to a cube, {5, 3} to a dodecahedron.

Q3 Explain why the entries in the first column must be at least 3. (Why are 0, 1, and 2 not possible?)

Q4 Explain why the entries in the second column must be at least 3. (Why are 0, 1, and 2 not possible?)

Q5 If there were a {3, 6} row, meaning 3-gons, 6 at each vertex, the result would not be three-dimensional. Explain. (Refer to the sum of the face angles at a vertex.)

Q6 There cannot be {3, 7}, {3, 8}, {3, 9}, and so on, polyhedra. Explain.

Q7 Why are {4, 4}, {4, 5}, {4, 6}, and so on, impossible?

Q8 Why are {5, 4,} {5, 5}, {5, 6}, and so on, impossible?

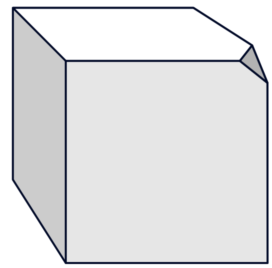
Q9 Why can't the first number be 6 or larger?

Q10 Summarizing: Explain why the only possibilities with less than 360 degrees at each vertex are the five polyhedra listed in the table.

3.4 Truncation

Challenge

Build a polyhedron, using blue and green struts, whose faces are equilateral triangles and regular octagons. (Hint: Start from a triple-size cube and truncate its corners to create new triangular faces.)



Cube with one corner truncated

Q1 The octahedron has four faces meeting at each vertex. What is the shape of the new polygon that appears when one of its vertices is truncated?

Q2 The octahedron has triangular faces. What is the shape of the polygon that takes the place of any triangle after its three vertices are truncated?

1. Build a triple-scale octahedron. (If you have green struts, build a regular octahedron with $3g_1$ edges; or, without green struts, build the red/blue approximation with $3b_1$ and $3r_1$ edges.)
2. Truncate one vertex of your triple-scale octahedron: First, add four struts that connect the balls at the one-third points adjacent to that vertex. Then remove the vertex and four struts connected to it. (The new face should correspond to the answer to Question 1.)
3. Truncate the other five vertices. (The polygons that replace the triangles should correspond to the answer to Question 2.) The result is the *truncated octahedron*.

Q3 Describe the number and type of faces in the truncation of each Platonic solid.

Q4 If a polyhedron *X* has *F* faces and *V* vertices, how many faces does the truncated *X* have?

4. Build a cuboctahedron. First, make a double-scale octahedron (regular or the red/blue approximation) so that there are balls at the edge midpoints. Then truncate all six vertices to the midpoints of the edges so that the new faces contact each other. (Add four struts around each vertex, and then remove the vertex and the four struts it contacts.)

Q5 Visualize the result of truncating a tetrahedron to its edge midpoints. What polyhedron results? If you have trouble with this, make a double-scale tetrahedron and truncate it to the edge midpoints.

Explorations 3

A. Cross Sections The cube has a regular 6-gon cross section. What is the largest regular *n*-gon cross section in each of the other Platonic solids? (Hint: Each can be constructed with a double-scale model.)

B. Dissecting Irregular Tetrahedra Construct an arbitrary irregular tetrahedron. Then make a double-scale model of it, and show how it can be dissected into an irregular octahedron surrounded by four smaller tetrahedra congruent to your original tetrahedron. Why is this possible from any initial tetrahedron?

C. Irregular Octahedra Construct an arbitrary irregular octahedron. Find a tetrahedron of which it is the inner core. Is this possible from any initial octahedron?

D. Parallelepiped Take a regular octahedron (or a red-blue approximation), and add two tetrahedra, on opposing faces, to form a solid with six parallelogram faces, called a *parallelepiped.* Since parallelepipeds can stack without gaps (like slanted bricks), can you see that octahedra and tetrahedra must somehow stack to fill space without gaps?

E. Plotting Polyhedra Another approach to the proof that there are only five Platonic solids is to make a plot in which the point (x, y) represents the polyhedron $\{x, y\}$, made of x-gons, y to a vertex. Plot the curve

$$y\left(180 - \frac{360}{x}\right) = 360$$

Where the curve has integer values, they correspond to flat constructions (360 degrees at each vertex); explain. The only possible regular polyhedra are the integer points below the curve, but with neither x nor y less than 3. Label the points for the regular polyhedra, and notice how symmetric the plot is. What does the symmetry correspond to?

F. Truncated Regular Polyhedra For each Platonic solid, *X,* construct a truncated *X* and/or an *X* truncated to its edge midpoints. Which starting solids lead to the same results?

G. Regular Skew Polygons It is easy to find skew 10-gons in a dodecahedron or an icosahedron. They are the zigzag "equators." In which Platonic solids can you find skew 6-gons? How about a skew 4-gon?

UNIT 4

Counting Strategies

It is important to be able to count the components of a polyhedron using logic and symmetry rather than counting one by one. The icosahedron is used to show two examples of counting strategies.

Goal

- To improve ability to visualize and count components of a complicated figure

Prerequisites

Units 1–3

Notes

This is a short, skills-oriented unit. Use the Challenge to set the stage, perhaps encouraging students to work in small groups. If students disagree with each other, encourage them to defend their answers with logical arguments. When the groups have reached agreement, or have failed to reach agreement, hand out Activity 4.1 and have students build the three models that are asked for at the beginning. As students begin to see patterns, such as that the number of faces in an icosahedron must be a multiple of five, logic will show them that an answer of 19 obtained by counting could not be right.

Note that the use of algebraic notation, starting in Question 8, is intended to encapsulate the understanding developed when working on the specific cases. It would be pointless to teach students those formulas up front, and it is not important for them to know these formulas by heart. They are easy enough to re-create when they are needed.

4.1 Counting Strategies

Challenge

Without looking at the actual polyhedra or your notes, make a table of the numbers of edges, faces, and vertices of each of the five regular polyhedra. Use logic to determine the numbers. It is OK to make sketches if that helps.

Here are logical methods for determining the number of edges, vertices, and faces for the icosahedron. Visualize an icosahedron model balanced on a vertex, and notice that there are five triangles around the top vertex, five around the bottom, and ten around the "equator," making a total of 20 faces. The ten around the equator can be seen as five up and five down (like the sides of an antiprism), so it should be immediately apparent that the number of faces is a multiple of 5.

Counting the icosahedron's vertices is even easier, because there are two groups of five, and also the north and south poles, making 12 at a glance.

Q1 Describe how to count the icosahedron's edges in groups of five.

Q2 Describe how to count the dodecahedron's edges in groups of five.

Q3 Resting a cube on a face, describe how to count its edges in groups of four.

Q4 Holding a cube up on a vertex, describe how to count its edges in groups of three or six.

Starting from the fact that there are 20 triangles, and each has 3 vertices, another method to count the icosahedron's vertices is to multiply, 20 times 3 equals 60. But since each vertex is shared by five triangles, vertices have been overcounted by a factor of 5. So divide, 60 divided by 5 equals 12, to get the answer.

Q5 Count the icosahedron's edges using this multiplication-division method. Be careful: How many triangles share an edge?

Q6 Use the fact that the dodecahedron has 12 faces, and count its vertices and edges.

Q7 Use the fact that the octahedron has eight 3-sided faces, meeting four to a vertex, and determine the number of vertices and edges.

Q8 If a polyhedron has n faces, and each face has k edges, how many edges does the polyhedron have? Check your formula on the Platonic Solids.

Q9 If someone tells you that there are exactly 17 faces on a certain polyhedron and that they are all triangles, what can you conclude?

Q10 If a polyhedron has n faces, each face has k edges, and each vertex is shared by d faces, how many vertices does the polyhedron have? Again, check your formula on the Platonic Solids.

Q11 Suppose a polyhedron has two types of faces. There are n_1 faces having k_1 edges and n_2 faces having k_2 edges. How many edges does it have?

Q12 If a polyhedron has v vertices and each vertex is an endpoint of d edges, how many edges are there altogether?

Explorations 4

A. Visualizing Edges Consider the complex polyhedra in Explorations 2, and count their edges systematically without building them. For example, the *elevated dodecahedron* is constructed by erecting a pentagonal pyramid on each face of a dodecahedron. One can visualize that this adds 12 times 5 equals 60 new edges to the 30 dodecahedron edges already present, making 90 edges total.

B. That's Odd For each of the polyhedra you have seen so far (the Platonic solids, pyramids, prisms), how many faces are there altogether with an odd number of edges (such as triangles and pentagons)? Is that number even or odd? Either build a polyhedron with an odd number of odd-sided polygon faces (for example, a polyhedron whose only faces are five triangles and any number of squares), or explain why it can't be done.

C. Handshakes At a party, various people shake hands when saying "hello" or "good-bye." Each handshake always involves exactly two people. By the end of the evening, some people have shaken hands an even number of times and some have shaken hands an odd number of times. Is it possible that an odd number of people shook hands an odd number of times?

D. Mixed Media The CD sculpture shown here consists of CD-ROMs facing alternately inward and outward, tracing the edges of a truncated icosahedron (12 pentagonal openings and 20 hexagonal openings). Count how many CDs are required to construct it.

Connection

Chronosynclastic Infundibulum,
34-inch diameter, CD-ROMs,
George W. Hart, 1998.

Source: Photograph courtesy of George W. Hart.

UNIT 5

Symmetry

Students locate the symmetry elements of the Platonic solids and other polyhedra, compare those to the symmetries of Zome models, and discover that different solids can have the same symmetries.

Goals

- To understand rotational and mirror symmetry elements
- To learn to identify and locate the symmetry elements of a polyhedron

Prerequisites

Units 1–4

Notes

If you don't have green struts, use paper models of the tetrahedron and octahedron. Or ask students to look at the symmetry of only the other three Platonic solids. Activities 5.2 and 5.3 require green struts.

For the purposes of this unit, it is sufficient that students develop a visual sense of rotational symmetry and reflection symmetry.

Review with your students the ideas of centers of rotational symmetry and lines of reflection symmetry in the case of plane figures. Then lead a short discussion of the symmetry of a regular polygon, when thought of as part of three-dimensional space. Students should see that a polygon has an axis of rotational symmetry passing through its center of rotational symmetry perpendicular to the plane of the polygon and that it has planes of reflection symmetry passing through the lines of reflection symmetry perpendicular to the plane of the polygon. In each case, going to three dimensions has added a dimension to the symmetry elements.

5.1 Icosahedron and Dodecahedron Symmetries

Challenge

Along how many planes can you cut an icosahedron in half and place it against a mirror to see it reconstructed? How many mirror planes do the other regular polyhedra have?

1. Hold an icosahedron lightly by two opposite balls so that you can spin it around, or stick a red strut on the outside of each of those two opposite balls to make an axis for spinning it.

This is a five-fold axis, meaning that if you spin it one fifth of a revolution, the icosahedron looks like it did in the original position. (It also looks unchanged if you spin it two fifths, or three fifths, or four fifths, or all the way around.)

An *n-fold symmetry axis* is an imaginary line with the property that rotating an object $^{360}/_{n}$ degrees about the line leaves it appearing unchanged.

2. In Exercise 1, you found that the line connecting a pair of opposite vertices of an icosahedron is a five-fold symmetry axis. There is a five-fold symmetry axis for each pair of opposite vertices. Two other types of symmetry axes for the icosahedron are described below.

   Three-fold axes: An imaginary line that connects the center of two opposite faces is a three-fold axis of symmetry. Spin your icosahedron around on a three-fold axis to see how a 120-degree rotation leaves it appearing unchanged.

   Two-fold axes: An imaginary line that connects the midpoints of two opposite edges is a two-fold axis of symmetry. Spin your icosahedron 180 degrees on a two-fold axis to see that it appears unchanged.

Notice that all the symmetry axes cross at the center. There are no other symmetry axes in an icosahedron; for example, there are no four-fold or seven-fold axes.

Q1 For the icosahedron, give the number of five-fold axes, three-fold axes, and two-fold axes.

Q2 In the icosahedron, if there are k n-fold axes, how are k and n related?

Q3 *True or False:* Any line through the center is a one-fold symmetry axis.

Imagine the plane that contains two opposite edges of your icosahedron. Each side of the plane is a reflection of the other side. If you could slice your model along this plane (cutting four of the balls in half and four of the struts in half) and put half against a mirror, it would look like the whole icosahedron.

A *symmetry plane* is an imaginary plane with the property that reflecting an object across it leaves the object appearing unchanged.

Q4 How many symmetry planes are there in an icosahedron?

The *symmetry of a polyhedron* means all of these symmetry elements and their arrangement relative to each other. Imagine a labeled line for each rotational axis, for example, labeled "5" for each five-fold axis. Imagine also a plane for each mirror plane. If you position these where they belong for a given polyhedron, and then erase the polyhedron, what is left is just your model of the symmetry, which you can study as an abstract object.

3. Spinning a dodecahedron model as necessary, find all the symmetry elements of the dodecahedron.

Q5 Describe the arrangement of the symmetry elements, relative to the dodecahedron's faces, vertices, and edges. How does this arrangement compare with that of the icosahedron?

Q6 For the dodecahedron, find the number of five-fold axes, three-fold axes, two-fold axes, and mirror planes.

Q7 How do the numbers of axes and mirrors in the dodecahedron compare to those in the icosahedron?

The icosahedron and dodecahedron have the same number and arrangement of corresponding symmetry elements, or the *same symmetry.* It is called *icosahedral symmetry.*

4. Use the Zome System to make a model of all the axes of icosahedral symmetry.

5.2 Simple Polyhedra Symmetries

Challenge

Take a cube and add some green struts to make an object with one four-fold axis and four two-fold axes, but no mirrors.

1. Build a prism on a regular 5-gon base.

   Q1 A right pentagonal prism has only one five-fold axis. Find its two-fold axes. Describe where they are. How many are there? (Hint: The two-fold axes do not go through the midpoints of opposite edges.) How many mirror planes are there? Describe where they are.

2. Build a prism on a regular 6-gon base.

   Q2 Describe the symmetry elements of a right prism on a regular n-gon base. Discuss the differences between the cases where n is even and n is odd.

3. Build an antiprism on a regular 5-gon base.

   Q3 A pentagonal antiprism, like a pentagonal prism, has one five-fold axis, but in other respects its symmetry is different. Find its two-fold axes and mirror planes. Describe where they are. How many of each are there?

4. Make a pentagonal pyramid and a square pyramid.

   Q4 Describe the symmetry elements of each pyramid.

5. Make a Zome model of a brick using four b_1s, four b_2s, and four b_3s. It should have six nonsquare rectangular faces.

   Q5 What are its symmetry elements?

6. Take a pentagonal prism. Using five blue struts, extend each of the edges of one 5-gon base, all clockwise. One end of these struts hangs unconnected, making a kind of "saw blade." Extend the edges of the other pentagon to make a saw blade in the opposite (counter-clockwise) direction.

   Q6 What symmetry elements does this object have, and how does it compare to the original prism?

Given any polyhedron that has axes and mirrors of symmetry, it is possible to make another object with the same axes of symmetry but no mirrors. One method is to attach some sort of n-fold spiral at each end of each n-fold axis. The object in Exercise 6 is one example, using prism symmetry.

7. Build, or find around you, the mirror image of the object in Exercise 6. (One approach is that you can simply look at it in a mirror.) In the mirror image, the clockwise and counterclockwise saw blades have switched positions.

An object that is different from its mirror image is called *chiral,* which means it comes in left-hand and right-hand forms. The two mirrored forms of a chiral object are called *enantiomorphs* of each other.

5.3 Cube and Related Symmetries

Challenge

The icosahedron and the dodecahedron have exactly the same symmetry. What other pair of regular polyhedra have a common symmetry? Explain.

Q1 Describe the symmetry axes and mirror planes of a cube. There will be two-fold, three-fold, and four-fold axes. How many of each are there, and where are they? (Note: If you use a Zome cube as a model, ignore the fact that in each square two blue struts are "up" and two are "flat." Pretend the blue struts are round.)

Q2 Describe the symmetry axes and mirror planes of a regular octahedron. How many of each are there, and where are they?

Q3 How do the cube and octahedron symmetries compare?

Q4 Describe the symmetry axes and mirror planes of a regular tetrahedron. How many of each are there, and where are they?

1. Using green struts, make a regular tetrahedron inscribed in a blue cube.

Q5 Describe how the axes of the tetrahedron and the axes of the cube are related. Describe how the mirror planes of the tetrahedron and the mirror planes of the cube are related.

2. Construct a $2b_1$ cube. Add a $2b_1$ edge to divide one face into two 1-by-2 rectangles. Divide the opposite square with a parallel edge. Divide the other four squares into two rectangles also, but choose the direction of the cuts so that the long side of each rectangle contacts the short sides of two other rectangles.

Q6 You have constructed what is often called a *pyritohedron,* because it has the same symmetry as crystals of the mineral iron pyrite. Describe the symmetry elements of the pyritohedron and how they differ from those of the cube.

Q7 If you ignore mirrors and look only at rotational symmetry, the pyrite symmetry is the same as what other symmetry?

Q8 Examine a Zome cube. What symmetry does it actually have if a blue strut lying flat is considered to be different from a blue strut resting on edge? In other words, consider the squares as having only two-fold symmetry, not four-fold symmetry, because a Zome square looks different after it is rotated 90 degrees.

Explorations 5

A. No Mirrors How can you modify a dodecahedron into an object with all of its rotational axes, but no mirror planes? Be careful, because extending each side of each pentagon with a saw blade exactly as in Question 6 of Activity 5.2 does not work. Why not? How can you modify the idea so that it does work?

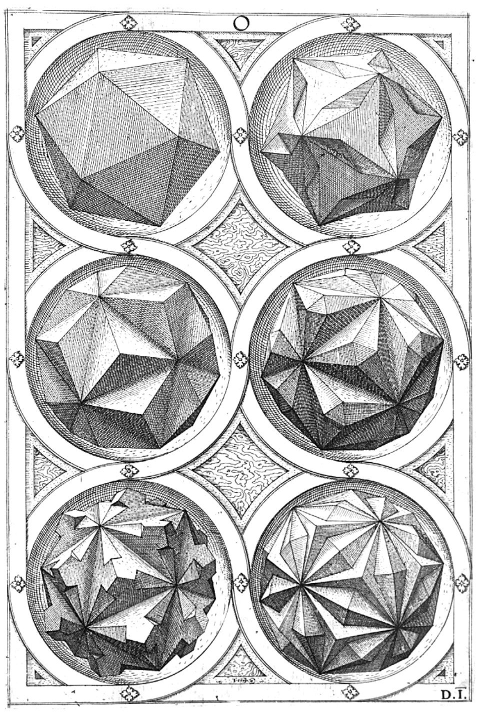
B. The Hexagon in the Dodecahedron How can you slice a regular dodecahedron to reveal a regular planar hexagon? Build a Zome model. How many such slices are possible? A larger model can show them all simultaneously.

C. 16-Hedron Construct a b_1 dodecahedron. Pick one vertex and connect its three neighboring vertices to each other with b_2s to make a b_2 triangle. Remove the chosen vertex and the three b_1s it touches. (As a result, you changed three pentagons into three trapezoids plus a triangle.) Rest the resulting structure on the triangle and look at the vertices in the top half: One vertex is at the top, three surround the top one, and six are in the next layer below that. Of those six, start anywhere and call the vertices 1, 2, 3, 4, 5, and 6 around in a circle. Mark vertices 1, 3, and 5 by putting an extra strut into one of their holes. Now treat each of the three marked vertices like the first chosen vertex: Surround it by a b_2 triangle and remove the marked vertex and its three b_1s. After all of this, you should have a 16-hedron consisting of 4 triangles and 12 isosceles trapezoids. (An *isosceles trapezoid* has its two nonparallel edges congruent.) What are its symmetry elements? (Hint: Find two-fold and three-fold axes.) Are there mirror planes?

D. Two Irregular Blue Icosahedra Take a b_1 dodecahedron and stand it on an edge in front of you. Six edges—the top, bottom, leftmost, rightmost, closest, and farthest edges—lie in the six planes of a cube. Add two diagonals to each dodecahedron face so that those six edges are each in two b_1-b_2-b_2 isosceles triangles. This also creates eight b_2 equilateral triangles. Remove all b_1s except the chosen six, and eight vertices will drop out. Notice that the result is an irregular icosahedron! Continue by removing the six remaining b_1s and adding six b_3s to make a concave irregular icosahedron. (Each pair of b_3s connects a pair of just-removed b_1s.) Both irregular icosahedra have the same topology as regular icosahedra: five triangles per vertex. Both have eight equilateral faces and twelve isosceles. What symmetry do these two irregular icosahedra have?

E. **Trapezoid-ohedron** This polyhedron has 24 congruent trapezoidal faces. Start with an r_2 rhombic triacontahedron (consisting of 30 r_2 rhombi; see E or F in Explorations 2). Choose six faces in the planes of a cube. (You can place it on a table with a face down, a face up, a face on the left, a face on the right, a face near you, and a face far from you.) Over just these six faces, build a rhombic pyramid, using two r_1s and two y_2s as slanting edges. Notice how coplanar triangles and rhombi combine into trapezoids; you can remove the dividing r_2 edges and combine $r_1 + r_2$ into r_3s. What symmetry does the result have?

Connection



This engraving by Jost Amman shows a series of objects with icosahedral symmetry, including one, at top right, which is chiral.

Source: Jost Amman, engraving from Wentzel Jamnitzer's Perspectiva Corporum Regularium, *Mark J. Millard Architectural Collection. Copyright © 2000 Board of Trustees, National Gallery of Art, Washington, D.C.*

UNIT 6

Euler's Theorem

Students consolidate all the data they have gathered on the edges, faces, and vertices of polyhedra and discover Euler's theorem. They learn to draw two-dimensional representations of the topology of polyhedra and use these drawings to solve a puzzle and to study Euler's theorem.

Goals

- To discover and apply Euler's theorem
- To apply algebraic manipulations representing polyhedral components
- To introduce a method for representing polyhedra in two dimensions

Prerequisites

Units 1–4 are suggested so that students will have experience with several examples of polyhedra.

Notes

Most of this unit does not use the Zome System directly, because the Zome System is designed around structured symmetric forms, and here we want to emphasize that Euler's theorem applies to any polyhedron, even asymmetric, unstructured ones.

6.1 Faces, Vertices, and Edges

In 1750, Leonhard Euler observed a simple fact about polyhedra that every mathematician for thousands of years before him had failed to notice: There is a simple formula that relates the number of faces, vertices, and edges of a convex polyhedron. If you know any two of those three numbers, you can use his formula to solve for the third. In this activity, students will discover Euler's theorem on their own. If they have trouble finding the formula, suggest that they focus on addition and subtraction.

Students should build only as much as is necessary to work out each answer. If they can visualize the polyhedra in their mind, or sketch them on paper, they don't need to build at all.

Note that while Euler's theorem holds for all convex polyhedra, it does not hold for just any structure. For example, an n-gon alone has $V = n,\ F = 1$, and $E = n$, which does not satisfy the theorem, but a polygon is not a polyhedron. (See Exploration A for more on this.)

The hint for Question 2 was covered in Unit 4. We will consider the case of some nonconvex polyhedra in Unit 10.

6.2 Topology

Questions 1–3 become increasingly difficult. The following hints may make this easier for your students.

- Start by drawing your outermost face.
- If a vertex is exactly opposite that face (as in 1), draw a point for it in the center of the diagram; if a face is exactly opposite (as in 2), draw it (small!) in the center of the diagram, paying attention to its orientation.
- Make sure that you preserve the number of edges out of each vertex. Remember that any polyhedron must have at least three edges meeting at every vertex.

Although at this level it is not a likely problem, it is possible to make a Schlegel-like diagram that does not represent a polyhedron, even with three or more edges out of each vertex. For example, draw the diagram for a cube twice, and then, with one more edge, connect one vertex of one cube to one vertex of the other.

Several students might collaborate to answer Question 4.

6.1 Faces, Vertices, and Edges

Challenge

For any polyhedron, let F, E, and V be the number of its faces, edges, and vertices, respectively. In 1750, the Swiss mathematician Leonhard Euler (rhymes with *boiler*) (1707–1783) discovered a simple relationship between F, E, and V that holds for all convex polyhedra. What is it?

A *convex* polyhedron is one with no indentations. An indentation is *concave.* One way to think of convexity is that the whole of a convex polyhedron lies on the same side of any given face plane. Euler's theorem holds for all convex polyhedra, but only some nonconvex polyhedra. So, in this unit, we will assume every polyhedron is convex.

1. In a table like this, consolidate your data for all the polyhedra you have built. For some of the rows, the entries are a function of n. Add more rows if you have explored other polyhedra.

Polyhedron	**Faces, *F***	**Vertices, *V***	**Edges, *E***
tetrahedron			
regular octahedron			
cube			
regular icosahedron			
regular dodecahedron			
triangular prism			
pentagonal prism			
n-gon prism			
triangular antiprism			
pentagonal antiprism			
n-gon antiprism			
square pyramid			
pentagonal pyramid			
n-gon pyramid			

Q1 Find a formula relating *F*, *V*, and *E*. Check that it holds in every row of the preceding table, including when you use the formulas involving *n*.

2. Invent your own new polyhedron, different from your neighbors' (for example, attach a pyramid to an antiprism or to another pyramid; be creative!). See if Euler's theorem holds for your new polyhedron. Is it convex?

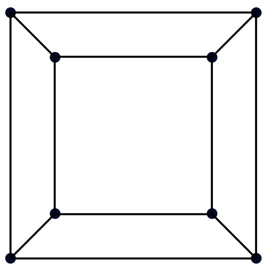
Q2 How many vertices are in a polyhedron consisting of *n* triangular faces? (Hint: A polyhedron consisting of *n* triangles has how many edges?) Verify your answer with some examples.

6.2 Topology

Challenge

You want to find a round-trip path along edges of an octahedron that visits every vertex exactly once. (Imagine a planet with 6 cities and 12 roads, arranged like an octahedron. You live in one city and want to visit all the others for vacation.) Use drawings so that you can try paths on paper. Repeat this puzzle on the edges of a dodecahedron.

Euler's discovery was the beginning of *topology,* a branch of mathematics that looks at only certain aspects of geometry. Topology is concerned with the number and connection of edges and vertices, but not the exact shapes, angles, or lengths. If rubber objects are bent and stretched (without tearing), then their geometric properties change, but their topological properties remain unchanged.



A cube laid flat

For example, the diagram shown here represents a cube that has been laid flat by stretching one face to be very large and adjusting angles at will. (Think of the very large face as the region outside the figure, an inside-out square going off to infinity.) This type of drawing preserves the topological properties of the original polyhedron (such as the value of V, F, and E, the number of edges that meet at each vertex, and so on). It is called a *Schlegel diagram* (after the mathematician Victor Schlegel, 1843–1908, who first used them in the 1880s). The two-dimensional format of Schlegel diagrams makes them useful for recording polyhedra on paper for studying their topological properties.

Q1 Draw a Schlegel diagram for

- **a.** a regular tetrahedron
- **b.** a pentagonal pyramid

Q2 Draw a Schlegel diagram for

- **a.** a pentagonal prism
- **b.** a pentagonal antiprism

Q3 Draw a Schlegel diagram for

- **a.** a regular octahedron
- **b.** a regular dodecahedron
- **c.** a regular icosahedron

Q4 Find a polyhedron with 8 vertices and 12 edges, other than a cube. You may sketch it, make a Schlegel diagram for it, or build it with the Zome System. (Hint: Starting with a simpler polyhedron, you can add 1 edge by adding a diagonal to any face that has 4 or more edges. You can add 1 vertex by dividing an edge into two shorter pieces, but that also adds an edge, and later you must connect the new vertex to some other vertex so that it doesn't have only 2 edges.)

Q5 Count the faces of the polyhedron in your sketch for Question 4. How does your count compare to your neighbors' counts? Comment.

Q6 How does the Schlegel diagram change when a pyramid is erected over an n-sided face? Sketch the change for a given n, and describe the change to the number of faces, edges, and vertices.

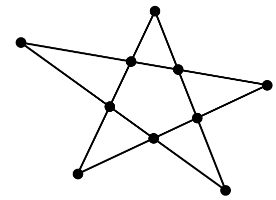
Q7 How is equality in Euler's formula preserved? (Why do both sides of the equation change by the same amount, when a pyramid is erected over an n-sided face?)

Q8 In general, how does the Schlegel diagram change when a vertex of a polyhedron is truncated? (Suppose there were n faces meeting at the vertex.) Sketch the change for a given n, and describe the change to the number of faces, edges, and vertices.

Q9 How is equality preserved in Euler's formula when truncating?

Explorations 6

A. Plane Figures Figures in the plane can also consist of faces, edges, and vertices. For example, the one shown here has 7 faces, 15 edges, and 10 vertices. It's not a polyhedron, because some of its vertices have only two edges. When looking at such figures, make sure you count all line intersections as vertices and count the entire exterior as one face. (In this example, the exterior face is a 10-gon.) There is a formula relating the number of faces, vertices, and edges of figures in the plane. Find it.



Plane figure

B. Try Truncation Find as many convex polyhedra as you can with 7 faces and 10 vertices. (A sketch or diagram suffices, but build them if you have time.) For each polyhedron, check that Euler's theorem holds.

C. Seven Faces Not every polyhedron with 7 faces has 10 vertices. Find polyhedra with 7 faces and 6, 7, 8, or 9 vertices.

D. Seven Edges Using Euler's theorem, you can prove that there is no polyhedron with 7 edges. Suppose that $E = 7$, and then find a contradiction. You saw in Unit 4 that if a polyhedron consists entirely of triangles, $E = (^3/_2)F$. Show more generally, since every face has at least three edges, that in any polyhedron $E \geq (^3/_2)F$. Conclude from this, since there must be at least four faces, that if $E = 7$, then $F = 4$. Make an analogous argument, since every vertex meets at least 3 edges, that if $E = 7$, then $V = 4$. Now use Euler's theorem to reach a contradiction, proving that the assumption $E = 7$ is impossible.

E. Platonic Solids Starting from Euler's theorem, we can prove algebraically that there are exactly five Platonic solids, without summing angles the way we did in Unit 3. The regular polyhedra $\{p, q\}$ consist of p-gons, meeting q at a vertex. By the counting methods of Unit 4, show that $qV = 2E = pF$. Call this number k; for example, $k = 60$ for the icosahedron and dodecahedron.

Solve these relations for $V, E,$ and F in terms of $p, q,$ and $k,$ and substitute into $V - E + F = 2$ to show that

$$V - E + F = 2 = k\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right)$$

Now, from the fact that $E = ^k/_2$, eliminate k and show that

$$E = \frac{2pq}{2p - pq + 2q}$$

(Or avoid using k. Instead, use $qV = 2E = pF$ to solve for V and F in terms of $E,$ and substitute in Euler's formula, then solve for E.) Since E and the numerator must be positive, the denominator must be positive.

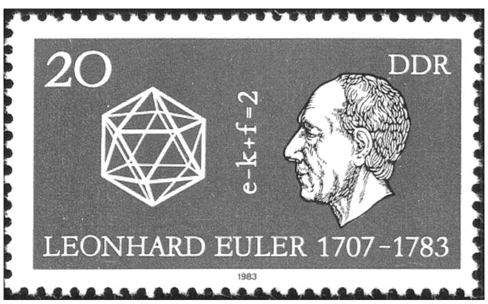
Show that $(p - 2)(q - 2) < 4$ is equivalent to $2p - pq + 2q > 0$. Since p and q must be at least 3, $(p - 2)$ and $(q - 2)$ are two positive integers whose product is less than 4. What are the possibilities for $(p - 2)$ and $(q - 2)$, and what are the corresponding regular polyhedra?

F. Fancy Formulas Exploration E shows how to find the number of edges of any platonic solid $\{p, q\}$ as an algebraic formula:

$$E = \frac{2pq}{2p - pq + 2q}$$

Find analogous formulas, using only p and q, for the number of vertices and faces in $\{p, q\}$.

Connection



Great mathematicians and their discoveries have been celebrated in many ways. This stamp from the former East Germany commemorates Euler and his theorem.

UNIT 7

Sum and Ratio Patterns

Students use both measurement and what they know about angle and side relationships in geometry to find patterns that relate the lengths of the Zome struts of the same color to each other. In the process, they will learn about the golden ratio.

Goals

- To understand the additive and multiplicative patterns embedded in the Zome struts
- To learn to recognize congruent angles by close observation of the zomeball
- To learn about the golden ratio and its use in scaling Zome models

Prerequisites

To complete Activity 7.1, students need to understand several theorems (listed in the Notes). For all the activities, they will need some facility in working with ratios and must be familiar with the quadratic formula, similarity, and scaling.

Notes

You can do either one or both of the first two activities, depending on the level of understanding and the background of your students. Green struts are used for the last part of Activity 7.2.

7.1 Finding the Patterns Using Geometry

Students must understand these theorems before they begin this activity:

- If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram, so both pairs of opposite sides are equal.
- If the corresponding angles of two triangles are equal, then the triangles are similar and the sides are proportional.
- A line parallel to one side of a triangle divides the other two sides into proportional segments.

The assumptions about the Zome System stated on the activity page are reasonable. If they weren't, it would not be possible to build large Zome structures. However, of the two assumptions, the one that breaks down more frequently is the assumption that struts that appear straight are indeed straight. As your students work, they are bound to encounter situations in which one or more bent struts provide a fallacious solution to a problem. However, that is not likely in this activity's constructions.

If students have trouble with the Challenge, you should be willing to go on with only one pattern for the struts discovered. For Questions 4 and 5, students will find that not every isosceles triangle they can make using the sum property will allow the extra zomeball(s) to be connected as the problem requires. For Question 4, a triangle with a b_2 base and one equal side of $b_1 + b_2$ will work if the b_1 strut is adjacent to the base. For Question 5, the equal side made up of $y_1 + y_2$ and the base of $b_1 + b_2$ must have the shorter part of the side next to the same vertex.

7.2 Finding the Patterns Using Measurement

This is a heavily guided activity, where students will not be exercising their problem-solving muscles. However, it provides younger students access to the ideas of the previous activity, and it may be worth doing even in some of the classes that did the first activity. Two reasons: It could give some concrete substance to the work done there; and it would allow students to do the final two exercises, which are a good way to assess their understanding of what's going on.

Accept any reasonable answers to the Challenge. This is a question that may be taken up again later on, for example, in Unit 13.

7.3 The Golden Ratio and Scaling

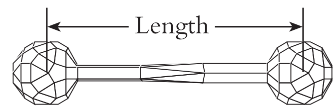
This is an exact calculation of the ratio approximated in the previous unit. The ratio appears in many contexts, both artistic and mathematical. In the shorter run, understanding the sum and ratio patterns of the Zome struts allows students to scale buildings in the ratio that relates one Zome strut to the others of the same color, and therefore to build ever-bigger constructions. This will turn out to be a critical skill in some of the future units.

You may want to precede this activity with some work on the Fibonacci sequence. (See Explorations D and E.)

7.1 Finding the Patterns Using Geometry

Challenge

Find two ways in which the lengths of the struts of the same color are related to each other. "The lengths of the struts" refers to the distance from the center of a ball on one end to the center of a ball on the other end.



Measurement of strut length

In all geometric proofs using the Zome System, assume that if the struts appear straight, they are straight, and that equal angles can be determined by careful observation of the ball at the vertices.

1. Make a parallelogram, using six struts, all different, but only red and blue. (A rectangle is acceptable.)

   Q1 Prove the figure you made is really a parallelogram. (Hint: Use angles.)

2. Make a parallelogram using six different yellow and blue struts.

   Q2 Prove the figure you made is really a parallelogram.

3. Make a parallelogram using six different red and yellow struts.

   Q3 Prove the figure you made is really a parallelogram.

   Q4 Explain how your parallelograms reveal the sum pattern: $b_1 + b_2 = b_3$, $r_1 + r_2 = r_3$, and $y_1 + y_2 = y_3$.

You have verified the *sum pattern*.

4. Use four blue struts and four balls to make an isosceles triangle that is not equilateral. Use the sum pattern when you construct the two equal sides. Connect the ball that is not a vertex of the triangle to the vertex it is not already connected to.

   Q5 Prove that the figure you made includes two similar isosceles triangles.

   Q6 Explain how it follows from similar isosceles triangles that $\frac{b_2}{b_1} = \frac{b_1 + b_2}{b_2}$ or $\frac{b_2}{b_1} = \frac{b_3}{b_2}$.

You have demonstrated the *ratio pattern*. This ratio is called τ (the Greek letter tau).

5. Use the sum pattern for the yellow struts to make an isosceles triangle with $b_1 + b_2$ as the base. Connect to each other the two balls that are not vertices.

Q7 Prove that the figure you made includes two similar isosceles triangles.

Q8 Explain how it follows from similar isosceles triangles that $\frac{y_2}{y_1} = \frac{b_2}{b_1}$ and $\frac{y_3}{y_2} = \frac{b_3}{b_2}$ and, therefore, that $\frac{y_2}{y_1} = \frac{y_3}{y_2} = \tau$.

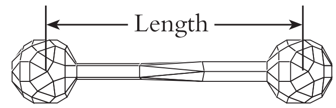
6. Use the sum pattern for the red struts to make an isosceles triangle, using $b_1 + b_2$ as the base. Connect to each other the two balls that are not vertices.

Q9 Prove that the figure includes two similar triangles and explain how it follows that $\frac{r_2}{r_1} = \frac{b_2}{b_1}$ and $\frac{r_3}{r_2} = \frac{b_3}{b_2}$ and, therefore, that $\frac{r_2}{r_1} = \frac{r_3}{r_2} = \tau$.

7.2 Finding the Patterns Using Measurement

Challenge

Explain how the mathematicians who designed the struts can determine if the engineers made them accurately.



Measurement of strut length

In this activity, you will use measurement to find patterns that relate the lengths of the struts of the same color to each other. "The lengths of the struts" refers to the distance from the center of a ball on one end to the center of a ball on the other end.

Q1 Measure the length of the struts as exactly as you can, to the nearest millimeter. Fill in this table, in millimeters:

Strut	b_1	b_2	b_3	y_1	y_2	y_3	r_1	r_2	r_3
Length									

Q2 Add the specified lengths to find a pattern.

$b_1 + b_2 =$ ______

$y_1 + y_2 =$ ______

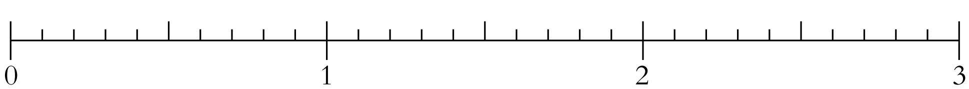
$r_1 + r_2 =$ ______

If your measurements and calculations were accurate, you should notice that the result in each case is approximately the same as a number you have already seen. Summarize the pattern you observe. (If you do not see a pattern in all three cases, recheck your measurements or sum.)

Q3 Do some divisions of your lengths. Use a calculator, but record results rounded to the hundredths place. Fill in this table:

What to divide	$\frac{b_2}{b_1}$	$\frac{b_3}{b_2}$	$\frac{y_2}{y_1}$	$\frac{y_3}{y_2}$	$\frac{r_2}{r_1}$	$\frac{r_3}{r_2}$
Your result						

Q4 Make a dot on the number line for each of your six division results.



You should find that all six dots are approximately in the same place. If not, recheck your divisions and measurements.

Q5 Summarize your results:

The length of a strut divided by the length of the next shorter strut of the same color is approximately _____.

If the Zome System were manufactured perfectly, if you could measure it perfectly, and if you made the calculations correctly, you would have discovered two patterns:

Sum pattern: $x_1 + x_2 = x_3$ (where x is **b**, **y**, or **r**)

Ratio pattern: $\frac{x_2}{x_1} = 1.62$ (approximately)

$\frac{x_3}{x_2} = 1.62$ (approximately)

Q6 Imagine that x_0 is a strut shorter than x_1 in the same color and that x_4 is longer than x_3 in the same color. Assume the ratio pattern continues in both directions. How long would those struts be?

b_0_______ b_4_______

y_0_______ y_4_______

r_0_______ r_4_______

Q7 Check whether the sum pattern would also continue. (Does $x_0 + x_1 = x_2$? Does $x_2 + x_3 = x_4$?)

Green Lengths There are different families of green lengths possible. In each family, the members scale by this same factor. You can find out which lengths you have by reference to the blue struts. Green struts that are the same length as blue struts are in the gb_1, gb_2, gb_3, . . . family. They make regular octagons with the blues of the same length. Green struts that are the diagonals of blue squares are in the g_1, g_2, g_3, . . . family. You can make a b_1–b_1–g_1 isosceles right triangle and a b_2–b_2–g_2 one. Because they are similar, you know g_2/g_1 is the same ratio τ as b_2/b_1. In this book, only gb_1, g_1, and g_2 are used.

7.3 The Golden Ratio and Scaling

Challenge

Find the exact value of the number that you measured to be approximately 1.62—the ratio pattern that you found in Activity 7.2.

The ratio, τ, between one strut and the next smaller one of the same color is called the *golden ratio.* It is the only number that makes it possible to have both the sum and ratio patterns. In this activity, you will prove this algebraically and find the exact value of τ.

Say that b_1 equals 1.

Q1 Use the ratio pattern to find b_2 and b_3 in terms of τ.

Q2 Use the sum pattern to write an equation.

If you answered these questions correctly, you should have a quadratic equation with unknown τ.

Q3 Solve the equation. Find both an exact answer and an approximate numerical value.

If you solved the equation correctly, you have found the golden ratio. (The negative root is not relevant when considering strut lengths.)

Q4 Using a calculator, find numerical values for

a. $1/\tau$

b. τ^2

For each, use addition or subtraction to write an equation that expresses it in terms of τ.

Consider the sequence $1, \tau, \tau^2, \tau^3, \ldots$. Each term is obtained from the previous term through multiplication by a certain factor.

Q5 What is the factor?

Since there is a common factor, the sequence is geometric.

Q6 We know that $1 + \tau = \tau^2$. Prove that it is true that $\tau + \tau^2 = \tau^3$ and, in fact, that $\tau^n + \tau^{(n+1)} = \tau^{(n+2)}$.

Q7 How can we make struts of length x_4 and x_5 using only the available struts x_1, x_2, and x_3?

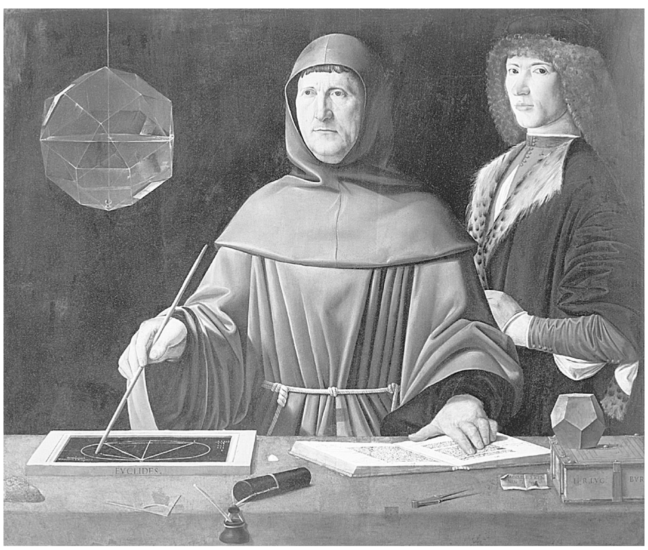
The sum and ratio patterns are useful for scaling Zome constructions.

1. Make a y_1–b_2–r_2 triangle. Now make triangles similar to this one, but larger. (However, do not use whole-number scaling factors.) Including the original, see if you can make at least three different sizes.

2. Take a polyhedron made with more than one type of strut, and create a larger or smaller version, using a scaling factor of τ or $1/\tau$.

Q8 Explain why we measured struts lengths from ball center to ball center. Why not measure the actual struts?

Connection



In 1509, the mathematician Luca Pacioli was so enthralled with the number τ that he wrote a whole book, *The Divine Proportion,* about its various properties.

Source: Portrait of Fra Luca Pacioli, painted by Jacopo de Barbari, 1495. Capodimonte Museum, Naples.

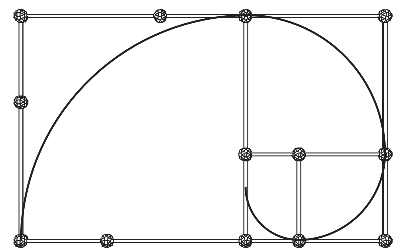
Explorations 7

A. Golden Rectangles A b_1-by-b_2 rectangle is called a *golden rectangle* because its sides are in the ratio τ. Build one, and then create a larger rectangle from it by using three b_2 struts to add a b_2 square on its side.

a. What is the ratio of the sides of the new rectangle?

b. Continue the process, building ever-larger golden rectangles.

c. How would you reverse the process? That is, starting from a golden rectangle, how would you find a smaller one inside it?


A golden spiral

By inscribing quarter circles in the squares, it is possible to draw a type of golden spiral, as shown in the figure above.

B. Five-Pointed Stars The 72-degree–72-degree–36-degree triangle is called a *golden triangle.* Build a b_1 regular pentagon, and add a golden triangle on each side. Then connect the new vertices, creating a larger regular pentagon. Repeat the process, building ever-larger nested five-pointed stars. What is the scale factor from one star to the next?

C. Smaller Struts Create a structure that positions two zomeballs at b_0 spacing. Repeat for r_0, y_0, $b_{(-1)}$, or $b_{(-2)}$. (Hint: Explorations A and B may help you get started.)

D. Fibonacci The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, ... It can be written $F_1 = 1$, $F_2 = 1$, and $F_i + F_{i+1} = F_{i+2}$. If we take the ratio of successive terms, we get the sequence $1/1$, $2/1$, $3/2$, ... Plot the ratio F_{i+1}/F_i as i gets larger. What value does it approach?

E. Other Starting Points A sequence can be called a *generalized Fibonacci sequence* if it is constructed with the formula $F_i + F_{i+1} = F_{i+2}$, using any two numbers as F_0 and F_1. (They do not have to be integers.) Choose starting numbers different from your neighbors', and find out what happens to the ratio F_{i+1}/F_i as i gets larger.

UNIT 8

Tessellations

Students build tessellations of one or more regular, irregular, or convex polygons, and they look at nonperiodic tilings. After students have learned about the angle relationships in tilings, they are asked to describe analogies they see between tessellations and polyhedra.

Goals

- To understand and create tessellations and tilings
- To learn concepts and notations important for understanding and discussing polyhedra

Prerequisites

Students should be familiar with the sums of the angles in triangles and other polygons and the angles of regular polygons (see Unit 1).

Notes

For a fuller introduction to tessellations, see Unit 7 in *Geometry Labs,* by Henri Picciotto, and the software *Kaleidomania,* by Kevin Lee (both available from Key Curriculum Press). The ultimate reference work on this subject is *Tilings and Patterns,* by Branko Grünbaum and G. C. Shephard.

8.1 Basic Tessellations

Students can get a substantial head start on this unit by spending some time on the Challenge. Acknowledge any discoveries, and have students share them with their classmates. Once no more progress is being made, hand out the activity page and have students start on the more guided part of the activity.

A key concept here is that, in a tessellation, the angles around a vertex must add up to 360 degrees. This helps answer most of the questions in this activity. (See the Answers for how.) However, while this condition is necessary, it is not sufficient. For example, the arrangement 10-5-5 works out around a single point but cannot be extended to tile the plane.

For one of the Archimedean tessellations, a Zome model requires green struts.

8.2 Nonperiodic Tilings

This activity is optional; it does not relate to the rest of this book in an essential way. Nevertheless, it is an interesting example of contemporary mathematics that the Zome System makes accessible. More information can be found in Martin Gardner's book *Penrose Tiles to Trapdoor Ciphers.*

8.1 Basic Tessellations

Challenge

Create some tessellations. Record them in your notebook. Can you find some with

- only one type of polygon?
- more than one type of polygon?
- regular polygons?
- nonregular polygons?
- nonconvex polygons?

A *tessellation* is a pattern of geometric figures that covers the plane and repeats infinitely in two dimensions, with no gaps and no overlaps.

Two well-known tessellations are the square pattern of the checkerboard and the hexagonal pattern of the honeycomb or chicken wire. Both patterns are widely used for tiling floors.

When you are building tessellations, two or three repetitions in each direction are enough to get the idea in each case. Do not spend too much time trying to extend any pattern off to infinity! You can create tessellations using the Zome System, or by using graph paper or a template with geometric figures if you have one. Or you can fold a piece of paper in half three times and cut out eight copies of the polygon you want to experiment with. The advantage of using the Zome System is that you can get equal angles and sides with great accuracy, and you can hold up your tessellations to show your classmates. The disadvantage is that not all angles and side lengths are available.

1. Find a scalene triangle tessellation.

   Q1 Sketch your tessellation, indicating the angles around a vertex.

2. Find a tessellation using an arbitrary quadrilateral.

   Q2 Sketch your tessellation, indicating the angles around a vertex.

3. Find a pentagon that will tessellate the plane.

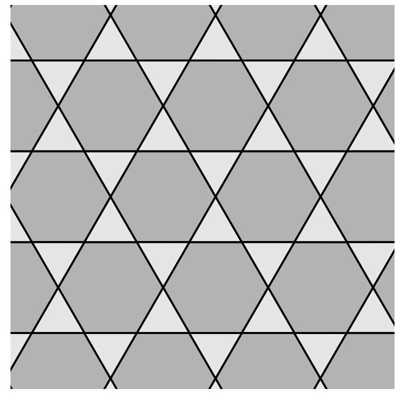
   Q3 Sketch your tessellation, indicating the angles around a vertex.

The simplest tessellations are composed of a single type of regular polygon. These are called the *regular tessellations.*

4. Find all the regular tessellations.

   Q4 Sketch your tessellations, indicating the angles around a vertex.

There are eight tessellations that involve two or more types of regular polygons, with the same arrangement at each vertex. These are called *Archimedean tessellations,* by analogy with the *Archimedean polyhedra,* which will be discussed in Unit 12. The one shown here is (3, 6, 3, 6), which says that each vertex is surrounded by alternating equilateral triangles and regular hexagons. It can be constructed with the Zome System, and so can (3, 3, 3, 3, 6). If you have green struts, then (4, 8, 8) can also be constructed, but to find others you will have to use some other tool. The zomeball does not allow them, since squares are in different planes than triangles and hexagons.



An Archimedean tessellation

5. Find some examples of Archimedean tessellations.

Q5 What analogies do you see between tessellations and polyhedra? What differences do you see?

8.2 Nonperiodic Tilings

Challenge

Fill the plane with polygons in a way that no combination repeats.

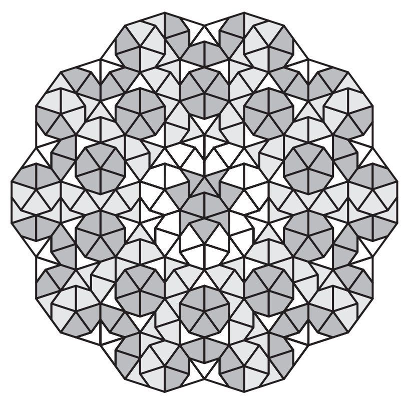
1. Find a parallelogram (or a rhombus) that can be used all by itself in more than one type of tessellation.

Q1 Notice that a regular hexagon can be dissected into three 60-degree rhombi. Does this suggest any new rhombic tessellations?

Tessellations work by exact repetition. But the plane can be filled in such a way that the pattern does not exactly repeat. This means that two different small regions might be the same, but if large enough neighborhoods are included around them, the large neighborhoods are always different. This is called a *nonperiodic tiling.*

One way to achieve a nonperiodic tiling is to use the tessellation of a regular hexagon as a starting point. Then you dissect a hexagon into three rhombi in two different ways. (Turn one 60 degrees to get the other.) If you start with the tessellation of regular hexagons and dissect them all into rhombi, each time choosing at random one of the two choices, large regions will never be exactly the same. There will be vertices with 3, 4, 5, and 6 edges.

In the 1960s, the British mathematician Roger Penrose (b. 1931) discovered ways to cover the plane with polygons in which the pattern is sure never to repeat, without needing to introduce randomness. A nonperiodic tiling of "kites and darts" is shown here. The Zome System allows these patterns.


Nonperiodic Penrose tiling

Q2 From the figure, determine the interior angles of Penrose's kites and darts.

2. Create a tessellation using Penrose's kites and darts.

3. Create a nonperiodic tiling using Penrose's kites and darts.

Penrose also discovered that two rhombi, with 36- and 72-degree angles, can be used to cover the plane in ways that do not repeat.

4. Create a tessellation using Penrose's rhombi.

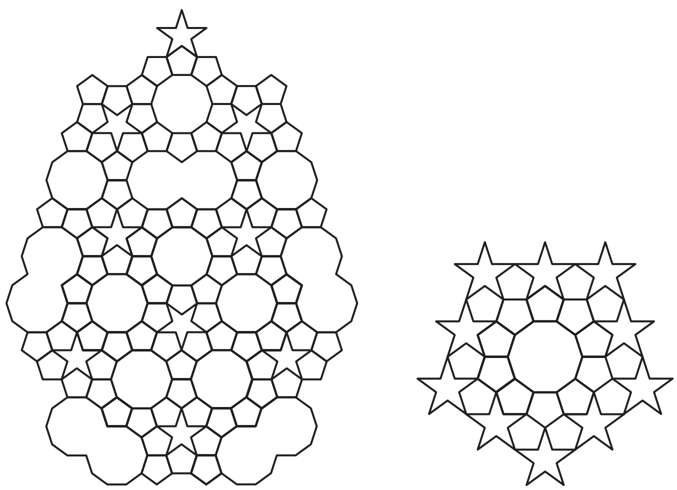
5. Create a nonperiodic tiling using Penrose's rhombi.

Explorations 8

A. Nonconvex Tiles Make a Zome tessellation with a nonconvex quadrilateral.

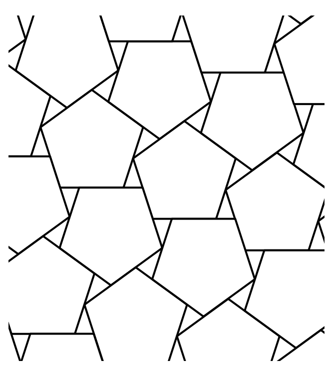
B. Pentagon Tiles Make a Zome tessellation that uses regular pentagons and just one other type of polygon. (The other type need not be regular.)

C. Kepler's Tessellations The German astronomer Johannes Kepler (1571–1630), who discovered that planets have elliptical orbits, was also interested in the problem of tessellations that involve pentagons. The figures replicate some patterns he published in the early 1600s, involving regular pentagons, regular decagons, and other polygons. Make one of these with the Zome System.



Copies of Kepler's tessellations

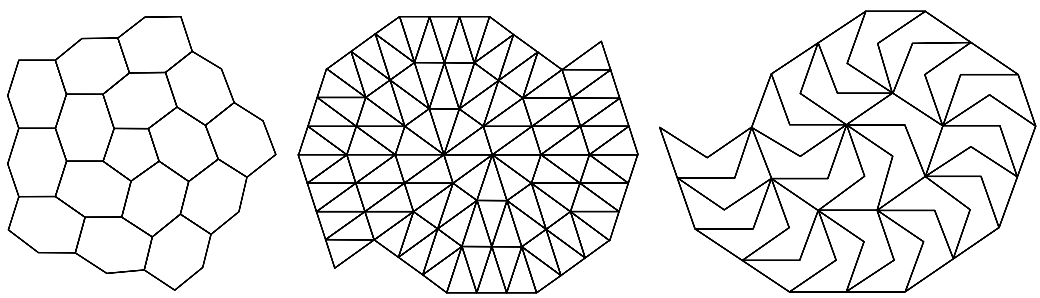
D. Pentagons and Triangles The figure below shows a pattern with regular pentagons and isosceles triangles. Each pentagon touches six surrounding pentagons. Make this pattern with the Zome System.



Pentagon and triangle tessellation

E. Escher M. C. Escher modified the edges of tessellations to make repeating human and animal forms that cover the plane. Explore by making your own Escher-like drawings, starting with any tessellation.

F. Spiral Tessellations The figures below show some interesting spiral tessellations. In the first, there is one regular pentagon surrounded by identical irregular equilateral hexagons. Explain how to extend it to infinity. The second is a double spiral composed of isosceles triangles. If you slide the bottom half to the left by the length of the side of the triangle, you would have a pattern with ten-fold symmetry. The third is a one-arm spiral, using a concave equilateral pentagon. Construct each of these with the Zome System.



Three spiral tessellations

UNIT 9

Duality

Students learn to find the dual of a tessellation. The dual to a given tessellation is another tessellation. It has a vertex in the center of each face of the given tessellation. They extend this concept to three dimensions and find duals of polyhedra.

Goal

- To understand the relationship between a tessellation and its dual, or between a polyhedron and its dual

Prerequisites

Unit 8, Tessellations, is necessary, and the ability to apply the Pythagorean theorem is needed for Question 7 of Activity 9.1 and for Activity 9.3.

Notes

9.1 Dual Tessellations

Tessellations provide an easy entry point into the concept of duality. Students are likely to be surprised by some of the dual tessellations. If there is a lot of interest in dual tessellations, encourage students to do Exploration A.

Question 7, calculating the needed lengths for the duals of tessellations, is not essential, but it does provide some practice with geometric and trigonometric ideas, and it throws light on the importance of having the right size of struts to make various constructions.

9.2 Dual Platonic Solids

Even after finding the dual of a tessellation, finding the dual of a polyhedron is difficult. If you give students the Challenge to find a dual polyhedron without the activity page, give the following hint: Scaling by a factor of 2 makes edges with a ball at the edge midpoint, where the edge of the dual polyhedron can cross it.

The first model in this activity requires green struts. To save time, have a few icosahedra pre-built. Or put the provided figure of the icosahedron on the overhead, and have a whole-class discussion.

The compound of the cube and octahedron is shown on the color insert as part of the model called the Rhombic Dodecahedron with Diagonals.

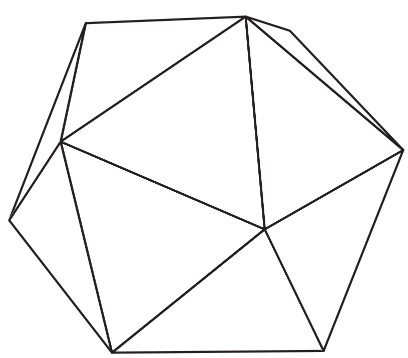
9.3 Dual Polyhedra

Question 3 is not essential, but it is a nice relationship and, again, a good opportunity to use geometry or trigonometry. It may be useful to review first, outside of a three-dimensional context, the theorem: Given a circle with center O and exterior point A, construct tangents AB and AC, and let D be the midpoint of chord BC. Then $OA/OB = OB/OD$. (Proof by similar right triangles.) As a special case, if it is a unit circle, then $OB = 1$ and OA and OD are reciprocal. In the three-dimensional context of this problem, A is a vertex of a polyhedron, D is the center of the dual's face, and B and C are the points where the edges of the original polyhedron and the dual are tangent to the unit sphere.

Another way to build a model to illustrate this result is to add a **$2y_2$** radius to the **$2b_2$** dodecahedron built in Activity 9.2 and hold a short red perpendicular as the slice through the icosahedron's face.

This book does not try to make a distinction between geometric duality and topological duality. Most students will not ask the question, but if it comes up, point out that given a polyhedron, only one polyhedron has the proper lengths and angles to satisfy the theorem in Question 3 and thus be a geometric dual. But the geometric dual can be "stretched" into an infinite number of forms that are still topologically dual to the original. For example, the blue and yellow Zome square pyramid is topologically dual to itself, but not geometrically self-dual.

One method to find the topological dual of a polyhedron is to build a pyramid on each face. Connecting the added vertices and removing the original polyhedron and the edges added when building the pyramids creates the topological dual of the original polyhedron, with each face replaced by a vertex and each vertex replaced by a face. In some cases, this can be done with the Zome System.



Icosahedron

9.1 Dual Tessellations

Challenge

Arrange two tessellations so that each has a vertex in the center of every interior of the other.

1. Make a tessellation of blue hexagons (at least six hexagons) and rest it flat on the table. For each edge of your tessellation, place a loose blue strut (the same size) on the table under it, at right angles to the strut it is under (midpoint under midpoint). If you lift up the original tessellation, you see a pattern of the unconnected blue struts. Do not connect them with zomeballs.

Q1 If the struts were connected, what kind of tessellation would they form?

The second tessellation is *dual* to the first. This means that each of its edges crosses an edge of the original tessellation at right angles. As you build tessellations and their duals, notice how a tessellation has one vertex at the center of each interior of its dual and vice versa.

2. Try the same thing as in Exercise 1, but start by constructing the tessellation of squares. Lay a loose strut under each edge at right angles. Lift up the tessellation, and see the pattern of the dual tessellation. Again, do not connect the struts with zomeballs.

Q2 If the struts were connected, what kind of tessellation would they form?

Q3 The square tessellation is *self-dual*. What do you think this means?

Q4 What is the dual of the tessellation of equilateral triangles?

3. Create the tessellation in which equilateral triangles and regular hexagons alternate (3, 6, 3, 6).

Q5 What is the dual of (3, 6, 3, 6)?

Q6 Make a conjecture about what happens if you start with any tessellation, take its dual, and then take the dual of the dual.

Q7 We did not try to connect the loose struts with balls because the struts are not exactly the right lengths (except for the case of squares). Calculate the needed lengths in terms of the lengths of the original struts:

a. For the hexagon tessellation

b. For the triangle tessellation

c. How are the answers to (a) and (b) related?

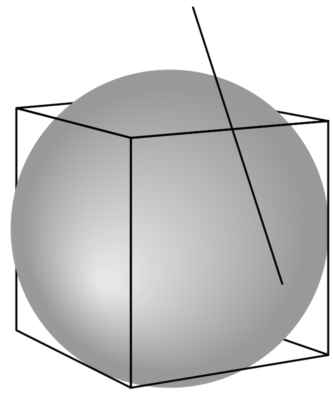
The most essential properties of duality are topological: Each tessellation has one vertex inside each interior of the other, and each edge crosses one edge of the other tessellation. An additional geometric property is that edge crossings are at right angles.

9.2 Dual Platonic Solids

Challenge

Given a Platonic solid, create another one so that all its vertices are directly above the centers of the faces of the original polyhedron. (Hint: Try double-scale.)

The ideas about duality for tessellations of polygons also apply to polyhedra, except that the patterns are bounded (wrapped around a ball) rather than on a plane that extends infinitely.



Cube with sphere inside and one dual edge

Q1 What is the dual to the cube? Imagine struts at right angles to each of the cube's edges. Instead of laying the struts flat on the table, imagine a sphere that touches the midpoint of every edge of the cube. Imagine laying the loose struts on the surface of the sphere, at right angles to the cube's edges, as in the figure.

Q2 What is the dual to the octahedron?

1. If you have green struts, make a model of the cube and octahedron together, showing their mutual duality. (Hint: Use double-length struts for the edges so there is a zomeball at the edge midpoint for the crossing.) This is *the compound of cube and octahedron in dual position.* Look at it for a while, focusing first on one then on the other.

Recall that the cube and the octahedron each have 12 edges. Now we see that this is not a coincidence, but a consequence of the fact that they are dual.

Q3 What can you say about the number of faces and vertices in the cube and the octahedron?

2. The dodecahedron and the icosahedron each have 30 edges. See if they are dual by trying to build a structure consisting of the two polyhedra, with their edges intersecting at right angles.

Q4 What can you say about the number of faces and vertices in the dodecahedron and the icosahedron?

Q5 If a polyhedron has k vertices, m faces, and n edges, what can you say about its dual?

Q6 If a polyhedron with V vertices, F faces, and E edges satisfies Euler's theorem, what can you say about its dual?

9.3 Dual Polyhedra

Challenge

Construct a pentagonal antiprism for which the ten triangular faces are equilateral. (This is just the middle section of an icosahedron.) Find its dual. You can just hold struts against it at right angles (remembering the imaginary sphere that the edges are all tangent to) or build a double-size model to make the compound of the antiprism and its dual.

Q1 Describe the dual of a pentagonal antiprism. (It could be called a *kite-ohedron*. Why?)

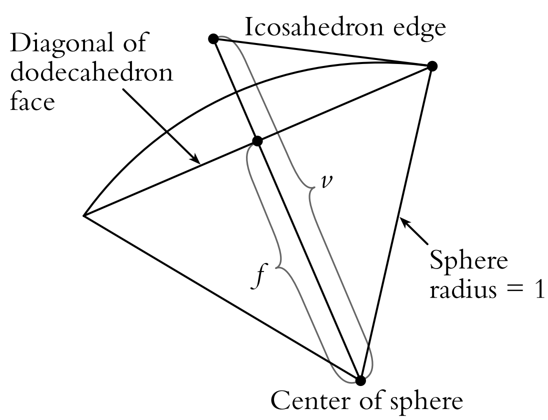
1. Make a $2b_2$ pentagon and raise it to a pyramid by adding $2y_3$ struts. This polyhedron will have the correct proportions to make a compound with its dual. (Hint: The yellow edges are divided at a point that is not their midpoints.)

Q2 What is the dual of a pentagonal pyramid?

2. Find a self-dual polyhedron you can build with green struts. (Hint: Recall the stella octangula.)

Q3 There is an elegant geometric relation between the dimensions of dual polyhedra when they are placed together with their edges tangent to a unit sphere. Note that each vertex of a polyhedron is directly above the center of a face of its dual. There is a simple formula relating the distance between each of these two points and the center of the sphere, which you can discover by analyzing a specific example.

The figure illustrates a slice through the compound of a dodecahedron and an icosahedron. The slice goes through one vertex of the icosahedron, one point where edges cross at right angles, and the center of the sphere. One face of the dodecahedron is perpendicular to the slice; the figure shows a slice of that face. Let f be the distance from the dodecahedron's face to the center of the sphere with a radius of 1, and let v be the distance from the icosahedron's vertex to the same center. Using similar triangles, derive a formula relating v and f.



Cross section through the compound of a dodecahedron and an icosahedron

This slice can be seen in a Zome model, except that one of the lines is not constructible. Start with the compound of the $2b_1$ dodecahedron with the $2b_2$ icosahedron. (It is not necessary to build the entire compound: One face of each polyhedron is enough.) Add one $2r_2$ radius from the center to an icosahedron vertex, and add a b_3 radius from an adjacent edge midpoint. Those two edges and the connecting icosahedron half-edge form the large triangle in the figure. The remaining line, corresponding to the slice through the dodecahedron's face, can be visualized by holding a y_2 with one end near the edge midpoint and the other end perpendicular to the red strut. (Do not connect this yellow strut, as it is not the correct length or angle.) Or you can hold the model so that you are looking right along the dodecahedron edge and the edge-on view of the plane of the dodecahedron face shows the missing segment.

The preceding figure is general and could represent the corresponding points and segments on any pair of dual polyhedra.

Q4 State the result from Question 3 in general terms, for any pair of dual polyhedra.

Explorations 9

A. Dual Tessellations Find the duals of some other tessellations from Unit 8, for example, (3, 3, 3, 3, 6) and (4, 8, 8).

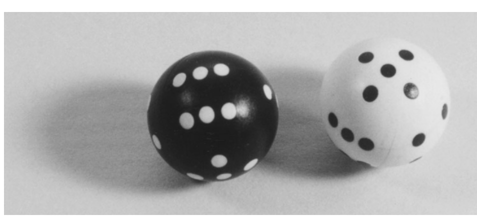
B. Rhombic Triacontahedron Dual Make the rhombic triacontahedron (see Explorations 2), and find its dual. Make a compound with its dual. The two polyhedra are different colors, making it easier to see them separately.

C. Kite-ohedra Find other *kite-ohedra* (polyhedra made of kite-shaped faces). One approach is to start with a skew polygon as its "equator," but be sure your faces are planar!

D. Prisms' Dual What is the dual to an *n*-gon prism? Sketch it. Can a prism's dual be made with the Zome System?

E. Spherical Dice The figure in the Connection below shows a pair of spherical dice, used like ordinary cubical dice, except that they have a spherical exterior. Like cubical dice, these dice have six "faces" marked with one to six dots. When you roll these dice, one of the six faces always lands clearly facing up, with equal probability. You can feel that they are hollow, with a small weight rattling loose inside. The shape of the inside cavity was designed so that it never lands partway between two numbers. What is the shape of the inside cavity that makes this work?

Connection



Based on the principle of duality, spherical dice that work like ordinary cube-shaped dice are possible. See Exploration E above.

Source: Photograph courtesy of George W. Hart.

UNIT

10 Descartes' Theorem

Students are given the definition of *angular deficit*. Then they draw on their knowledge of the angles of polyhedra and discover Descartes' theorem. Students are given examples of nonconvex polyhedra for which Descartes' theorem and Euler's theorem hold, and some for which the theorems do not apply.

Goals

- To discover Descartes' theorem
- To review Euler's theorem
- To use algebraic manipulation to represent polyhedral components
- To develop an understanding that both theorems hold for all convex polyhedra, but for only some nonconvex ones

Prerequisites

Units 1–4 are necessary, because students need to know enough examples of polyhedra to do this unit. Some of the ideas in Unit 8 would also be helpful, because of that unit's emphasis on the sum of angles around a point.

Activity 10.2 on nonconvex polyhedra involves Euler's theorem. To do it, students need to have done Unit 6.

Notes

René Descartes was a French mathematician and philosopher. Among other things, he is well-known for having invented the concept of coordinates (which is why they're called *Cartesian coordinates*) and for having said, "I think, therefore I am." In this unit, students learn about a theorem he discovered about polyhedra.

10.1 Angular Deficit

The Challenge could be done with other colors, but we won't know those angles until Unit 13.

The definition of *convex* is repeated here in case your students did not do Unit 6 on Euler's theorem.

Filling out the table is a substantial undertaking. Note that for each of the examples in it, all vertices are identical to each other. More complicated examples are to be found in Exercises 2 and 3. For the prisms assume rectangular faces, and for the antiprisms assume equilateral triangles, connecting the top and bottom *n*-gon. However, the result would still hold in other cases.

The examples involving variables (*n* or *x*) involve a bit of algebraic manipulation and may end up taking a long time for some students, but they are definitely worth doing. Students should find the final outcome of each calculation quite satisfying.

The frequency 2 icosahedral dome with triangles divided into four parts is not strictly convex, because along some edges two faces meet in the same plane. It is not concave either. Usually, it is assumed that no two faces of a polyhedron are in the same plane. Technically, *convexity* doesn't guarantee this, so *strict convexity* is assumed.

We will see in Unit 24 that a proof of Descartes' theorem follows directly from a proof of Euler's theorem.

10.2 Nonconvex Polyhedra

The notion of "simply connected polyhedra" is not simple. For an excellent discussion, see *Proofs and Refutations,* by Imre Lakatos. The examples presented here should provide enough of a warning to your students about the limits of Euler's and Descartes' theorems.

All polyhedra in this activity are nonconvex, so that is not the essential factor determining when the theorem applies. It is not easy to characterize the exact set of polyhedra for which the theorem applies. Intuitively, the first is somehow like a normal polyhedron just indented, while the other three are fundamentally abnormal in some sense. Mathematically, one says that the normal polyhedra, whether or not indented, are *simply connected,* while the last three examples are *not simply connected.* A simply connected polyhedron is one in which any loop of thread lying on its surface can be shrunk and slid around to gradually shrink to any point on its surface. The donut is not simply connected because a loop going around the hole or through the hole cannot be shrunk to a point. The double cube and hollowed cube are not simply connected because a loop on one part of the surface cannot be gradually shrunk to a point on the other part. The two theorems apply to simply connected polyhedra, which includes all convex polyhedra, but only some indented polyhedra.

To simplify this, most texts just state the theorems "for any convex polyhedron." They are correct, but they leave out the many nonconvex cases in which the theorem still holds. Euler himself, and many mathematicians after him, did not consider cases such as these three.

10.1 Angular Deficit

Challenge

If you place three blue struts in a zomeball, you will have created three angles. Do this in different ways, keeping track of the sum of the three angles. What is the largest sum possible? How do you get it? What is the smallest? How many other sums can you find?

A *convex* polyhedron has no indentations. An indentation is *concave.* One way to think of convexity is that the whole of a convex polyhedron lies on the same side of any given face. In this activity, you explore a theorem discovered by the French mathematician René Descartes (pronounced "day-CART"; 1596–1650). The theorem holds for all convex polyhedra, but only for some nonconvex polyhedra. Assume every polyhedron in this activity is convex.

Think about the difference between a polyhedron and a tessellation. In a tessellation, at each vertex the angles sum to 360 degrees, because it lies in a plane. In a convex polyhedron, at any vertex the angles sum to less than 360. Consider, for example, the cube, which has three squares meeting at each vertex. Three 90-degree angles sum to 270 degrees, which is 90 degrees less than 360 degrees. The 90-degree shortage is called the *angular deficit* at each vertex. Descartes' theorem is about angular deficit.

The *angular deficit* at a vertex of a polyhedron is 360 minus the sum of the face (interior) angles that meet at that vertex.

Q1 What is the angular deficit at any vertex of a regular icosahedron?

The *total angular deficit of a polyhedron* is the sum of the angular deficits at each of the vertices of the polyhedron. For instance, in a cube there are 8 identical vertices with a deficit of 90 degrees each, so the total angular deficit of a cube is 8 times 90 equals 720 degrees.

Q2 What is the total angular deficit of a regular icosahedron?

1. Fill out a table like this. (Build only as much as is necessary to work out each answer. If you can visualize the polyhedra in your mind, or sketch them on paper, you don't need to build at all.)

Polyhedron	Number of vertices	Angular deficit	
		At each vertex	Total
cube	8	90	720
regular icosahedron			
regular octahedron			
regular dodecahedron			
regular tetrahedron			
triangular prism			
pentagonal prism			
n-gon prism			
pentagonal antiprism (with equilateral sides)			
n-gon antiprism			

Q3 Make a hypothesis about the total angular deficit of any polyhedron.

If you answered the question correctly, you have stated Descartes' theorem. In the rest of this activity, you will verify that it applies to some other convex polyhedra.

2. Build a $2b_1$ equilateral triangle, and connect the three edge midpoints together, to make a $2b_1$ triangle composed of four b_1 triangles. Assemble five of these units as in a $2b_1$ icosahedron, and imagine extending it to a complete icosahedral form made of 20 times 4, or 80 equilateral triangles.

If a structure like that is used for a geodesic dome, it is called a *frequency 2 icosahedral dome.* The 2 means that each icosahedron edge is divided in half.

Q4 What is the total angular deficit of the icosahedral structure in Exercise 2, if completed? Notice there are two different types of vertices, with two different deficits. You still sum over all the vertices to get the total.

Q5 What happens with a frequency n icosahedral dome?

3. Build all or part of a rhombic triacontahedron. It is composed of 30 red rhombi. The acute angles meet in groups of five at 12 vertices, and the obtuse angles meet in groups of three at 20 vertices.

Q6 Call the acute angle of a red rhombus x.

a. What is the obtuse angle of the red rhombus in terms of x?

b. What is the total angular deficit of the rhombic triacontahedron? Simplify your answer.

Q7 Verify Descartes' theorem for a pyramid on a pentagonal base with five congruent isosceles triangles. Let each triangle have two base angles x.

a. What is the third angle of the isosceles triangles?

b. What is the total angular deficit for the pyramid?

Q8 Verify Descartes' theorem for a pyramid on an n-gon base with n isosceles triangles.

10.2 Nonconvex Polyhedra

Challenge

Build a polyhedron with a hole through it like a donut.

Although Descartes' theorem focuses on angles and Euler's theorem ignores them, these two theorems are actually closely related. Both deal with polyhedra as a whole, and, as you will see in this activity, they both fail to hold in certain nonconvex cases. In working on this activity, you might conclude that we have not been exact enough about the definition of a polyhedron.

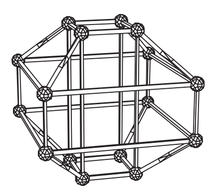
1. Build (part of) a dodecahedron and erect a blue pentagonal pyramid on the inside of each face. You will be adding a new vertex in the middle of, and slightly inside of, each dodecahedron face. When done, you have a new polyhedron consisting of 60 equilateral triangles, called a *nonconvex equilateral hexecontahedron.*

It is just a regular dodecahedron with each pentagon replaced by a concave dimple of five equilateral triangles.

Q1 Do Descartes' and Euler's theorems hold for this polyhedron?

A polyhedron with a hole through it, like a donut, is not convex.

2. Construct the polyhedron in the figure. It has a square tube running through it, and eight faces are trapezoids.


Donut-like polyhedron

Q2 Does Euler's theorem hold for this polyhedron?

Q3 Does Descartes' theorem hold for this polyhedron? In the trapezoids, there are two blue-yellow angles whose size we have not yet determined, but they are supplementary. The algebra of Descartes' formula works out very neatly if you call these angles $90 - x$ and $90 + x$.

If two cubes just touch at a vertex, is the combination one polyhedron? Different authors choose different definitions and don't always think about such marginal cases, so there is no single answer. If the definition of a polyhedron is just "a three-dimensional shape bounded by polygons," then it satisfies the definition.

Q4 Consider a two-cube object formed by joining two cubes at a common vertex. Does this object satisfy both theorems?

Q5 Consider an object formed by taking a solid cube and hollowing out a small cube-shaped interior cavity. The result is a region of space bounded by 12 squares. Does this object satisfy both theorems?

Of the four examples in this activity, you should have found that both theorems apply to one of them, but that both fail in the other three cases.

Explorations 10

A. Dimples and Donuts Create your own examples of a polyhedron that is nonconvex because it has indentations, and a donut-like polyhedron. Verify that Euler's and Descartes' theorems apply in the first case but not the second.

B. Soccer Ball Consider the truncated icosahedron, the shape of a soccer ball. At each vertex there is one pentagon and two hexagons. Using Descartes' theorem, determine how many vertices it has. Then construct it or part of it with the Zome System to verify your answer.

C. Triangle, Square, and Pentagon? Can there be a polyhedron in which an equilateral triangle, a square, and a regular pentagon meet at each vertex? (This problem is about all polyhedra, not just the ones that can be built with the Zome System.)

D. Sphere-like Polyhedra Descartes' theorem implies that sphere-like polyhedra must have many vertices. For a polyhedron to be sphere-like, each vertex must be almost flat so that its angular deficit is low. To have a total deficit of 720, there must be many such vertices. Conversely, if there are many identical vertices, each must have a low deficit so that the vertices are almost flat. There is a Zome-constructible polyhedron that has 120 identical vertices. Each face is a regular polygon, and at each vertex three polygons meet. Which three regular polygons must they be? Construct the polyhedron.

E. Pentagons and Hexagons Suppose a polyhedron consists entirely of regular pentagons and regular hexagons. Descartes' theorem determines the number of pentagons. Let the number of pentagons be F_5 and the number of hexagons be F_6, so $F = F_5 + F_6$. Assuming the faces are regular, how many edges must meet at each vertex? Find a formula for V in terms of F_5 and F_6. Another way of stating Descartes' theorem is that if we sum all the face angles at all the vertices, we get $360V - 720$, so

$$(6)(120F_6) + (5)(108F_5) = 360V - 720$$

Combine these results and determine F_5.

UNIT 11

Cubes in a Dodecahedron

Students explore the relationship between a cube and a dodecahedron by identifying the five cubes that can be inscribed in a dodecahedron. Students then create tetrahedra inside a cube and a dodecahedron.

Goals

- To learn about the relationships between the cube and the dodecahedron
- To see how tetrahedra are related to the cube and the dodecahedron

Prerequisites

Units 1 and 2 are essential so that students are familiar with the Zome cube and dodecahedron. They also need to be familiar with simple geometric proof. The symmetry questions require Unit 5. And familiarity with Unit 7 is needed to answer Question 5 of Activity 11.2.

Notes

Students can work together to build these aesthetically pleasing and memorable models. Because the models are rather large, it is important to remind students about basic Zome building principles: Push the struts all the way in; never push simultaneously on distant parts of the model; if a ball is needed at the end of a strut, it is best to add that ball before the strut is added to the larger figure.

11.1 One Cube

Students will be surprised to find the cube in the dodecahedron.

11.2 Five Cubes

As students build Model 3, suggest that in tight spots they leave the longer struts for the end, since these can be flexed to be put into place.

11.3 Related Constructions

This activity requires green struts. The compound of five octahedra suggested at the end of the activity is built in Explorations 22.

Explorations 11

Explorations B and C take a long time and a lot of pieces. (See the index of polyhedra for strut counts.)

11.1 One Cube

Challenge

Inscribe a Zome cube inside a b_1 or b_2 dodecahedron so that every vertex of the cube is a vertex of the dodecahedron.

Q1 Write down the counts of faces, edges, and vertices of the cube and the dodecahedron. Look at the number appearing twice. This is not a coincidence.

1. Make a b_1 or b_2 dodecahedron.
2. Find four of the dodecahedron's balls that lie at the vertices of a square. Note: The edges of the square are not there (yet), but the dodecahedron holds the vertices in space at the correct positions to make a square. Feel your imaginary square by holding the dodecahedron with a thumb or forefinger on each one of the four balls making up the square. If you are working with other students, pass the dodecahedron to them so that they can find other squares.
3. Insert four blue struts in your model to show the square's edges. (Use b_2s if it is a b_1 dodecahedron, or use b_3s if it is a b_2 dodecahedron.)

Q2 Use geometry to prove your square is really a square. (How can you tell that all four sides of your square are the same lengths and that the angles are each 90 degrees?)

4. Continue making a cube, using the square you have already made as one of its faces. Do not add new balls to the dodecahedron; use balls already present in it.

The cube and the dodecahedron are *concentric;* that means they share the same center.

Observe that the relationship between the number of pentagons in the dodecahedron and the number of edges in the cube is not a coincidence. There is a one-to-one relationship: Each cube edge is a diagonal of a different pentagonal face.

Q3 Notice that the three-fold axes of the cube align with three-fold axes of the dodecahedron, but the dodecahedron has other three-fold axes that do not align with the cube's structure. What other sets of axes do align between the cube and the dodecahedron?

11.2 Five Cubes

Challenge

Can you fit more than one cube into the dodecahedron? If so, how?

Q1 In making your cube, you found six squares whose vertices were also the dodecahedron's vertices. How many different squares can be found this way? (Hint: The center of each square lies directly behind what?)

1. Point out some other squares, not in your cube, using your thumbs and forefingers.

Q2 How many different cubes can you find in the dodecahedron?

You will make a model of all the cubes in the dodecahedron intersecting each other, but it will take some planning.

Q3 If you try to add a second cube to your dodecahedron, you will run into a problem. What is the problem? (If you do not see it right away, try holding a b_2 or b_3 between each of the 12 different pairs of balls that would be connected by the edges of your second cube.)

Each cube creates one diagonal in each of the pentagons of the dodecahedron. If all five cubes were there, then you would see all five diagonals for each pentagon of the dodecahedron.

Q4 Sketch on paper the shape you get by drawing all five diagonals of a regular pentagon.

2. Make a b_3 pentagon and then construct the diagonals inside the pentagon with five b_1s and with ten b_2s. (Use five balls to make connections where the diagonals cross.)

The five-pointed shape you just created is called a *pentagram*.

3. Construct the b_3 dodecahedron with all the diagonals of all the faces.
4. Remove the b_3 edges, and you have just the five intersecting cubes.

You have built the compound of five cubes.

Q5 Explain why the cubes' edges are of length b_4.

Q6 Which of the many axes of symmetry of the individual cubes align with each other or the various axes of symmetry of the dodecahedron?

11.3 Related Constructions

Challenge

Create a compound of five concentric regular tetrahedra.

1. Create a regular tetrahedron inside a cube. Green struts are required.
2. Create a regular tetrahedron in a regular dodecahedron. Green struts are required.
3. Create five tetrahedra in a regular dodecahedron. (Hint: No scaling is necessary.) (When your model is complete, you can remove all the blues except one 5-gon as a base for the compound of five tetrahedra.)

   Q1 In how many different ways can five tetrahedra be inscribed in a dodecahedron? No two tetrahedra should share a vertex.

If you take the dual of the two parts to the cube-in-a-dodecahedron, you get an icosahedron-in-an-octahedron.

4. Build a $2b_1$ icosahedron. Rest it on an edge and pick six of its edges that are perpendicularly arranged: top, bottom, left, right, front, and back. Mark these six edges by inserting a b_1 directly outward from the node at the edge midpoint. This creates two b_1 right angles; insert their g_1 hypotenuses. Add g_2s that extend these edges into a regular octahedron. Finally, remove the six b_1s that were used only for marking and scaffolding.

In the cube-in-a-dodecahedron, the eight vertices of the cube are eight vertices of the dodecahedron. In this dual model, eight of the faces of the icosahedron lie in the planes of the eight faces of the octahedron. This happens because vertices transform to faces when you take the dual. Dualizing also turns things "inside out" in the sense that the cube on the inside dualizes to the octahedron on the outside.

Q2 What does the tetrahedron-in-the-dodecahedron dualize to?

5. Build the dual to the tetrahedron-in-the-dodecahedron.

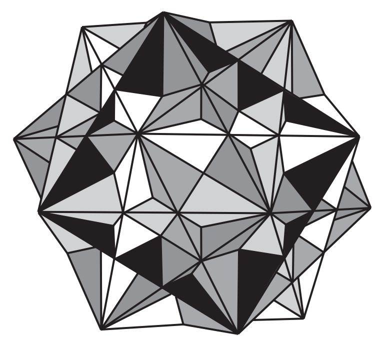
The dual to the five cubes is a compound of five octahedra surrounding one icosahedron. You can extend your octahedron-around-an-icosahedron by adding four more $g_1 + g_2$ octahedra, but the g_2s need to bend a bit around each other. It is also possible to scale up, so as to have a node at the crossing point, but a great many pieces would be needed.

Explorations 11

A. Roof Shapes The ancient Greeks discovered the cube-in-the-dodecahedron over 2000 years ago. In Euclid's classic text on geometry, *The Elements*, compiled around 300 B.C., he makes this construction, but going in the opposite direction. Euclid starts with a cube and adds six roof shapes to form the dodecahedron. Try to build a dodecahedron that way. (A roof shape is made of five pieces of the same length; when added to the side of a cube, the roof shape contains two triangles and two isosceles trapezoids.)

B. Intersection of Five Cubes The common space inside all cubes in the compound of five cubes is a rhombic triacontahedron (a polyhedron with 30 rhombus faces, built from red struts in Explorations 2). If your cubes are $b_2 + b_1 + b_2$, their intersection is an r_2 rhombic triacontahedron floating inside. Its vertices need to be connected to the cube's edges using some scaffolding, such as y_2s to the cube's vertices or r_2s to the pentagram's crossing points. Add the common intersection to your model.

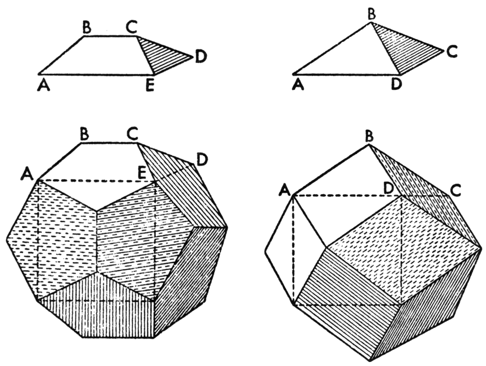
C. Faces of the Five Cubes Zome models are vertex and edge models, which do not show face planes. This figure shows the faces of the five cubes and how they cut through each other. Wherever two planes cross is a line of intersection. Add some of these lines of intersection to your five-cube model. They are red and yellow. You can tell visually if you have placed them correctly because you can sight along a square to test that a red or yellow strut is in its plane, that is, not sticking out above or below the plane. The line of intersection of two planes is the only line that passes this test from both planes. Note: If your cubes have $b_2 + b_1 + b_2$ edges, some segments of these intersection lines would require r_0 struts, so scale everything up another size to b_5 cubes if you want to be able to show even the smallest segments of this model. The inner r_3 rhombic triacontahedron is then very apparent.



Compound of five cubes

D. Rhombic Dodecahedron There is another kind of dodecahedron associated with the cube. The six roofs on the cube that make the dodecahedron can be replaced with six yellow square pyramids. Doing so makes a yellow *rhombic dodecahedron.* It is a dodecahedron because there are 12 faces, but now each face is a yellow rhombus.

The astronomer and mathematician Johannes Kepler discovered this shape in the 1600s. The figure shows his drawings of both dodecahedra around a cube.



Dodecahedron and rhombic dodecahedron around cubes

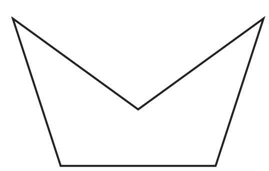
Source: Johannes Kepler, Epitome of Copernican Astronomy & Harmonies of the World. *Copyright ©1995. Reprinted by permission of Prometheus Books, New York.*

Make a rhombic dodecahedron and notice the one-to-one relationship between the edges of the cube and the short diagonals of the rhombi. Remove the blue struts to have just the rhombic dodecahedron.

If you build a rhombic dodecahedron around each of the five cubes in the compound of five cubes, you get a compound of five rhombic dodecahedra. This is a truly beautiful construction but needs 240 y_2s to complete it in size y_4, with $y_2 + y_1 + y_2$ edges. It is about a meter in diameter in size y_5.

E. Octahedron and Icosahedron Activity 11.3 showed how to construct an icosahedron-in-an-octahedron, with every icosahedron vertex on an octahedron edge. Can you reverse this order and use green struts to build a regular octahedron inscribed in a regular icosahedron, with all the octahedron's vertices on some icosahedron edge?

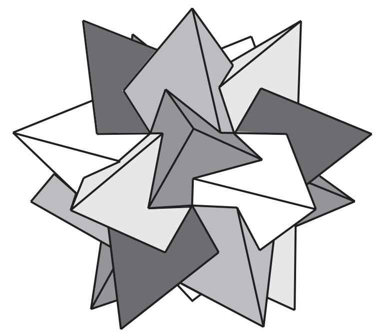
F. Concave Dodecahedron There is another kind of pentagonal dodecahedron that you can make. The ordinary regular dodecahedron has a regular pentagon for each face. The new dodecahedron is called a *concave dodecahedron* because it has a concave pentagon for each face. If you take a paper pentagon and fold it over on a diagonal, the five edges form a concave pentagon, as in this figure.



Concave pentagon

Make a concave pentagon from five b_1s. It is possible to assemble twelve copies of this shape into a concave dodecahedron. Here is one method: Start with the cube in the dodecahedron and think of it as Euclid did, as a cube with six roofs added. Then subtract the six roofs from the cube; that is, move the six roofs to the inside of the cube's faces. Study the result to see that it has twelve faces, each a concave pentagon. After you build it, look for a b_0 icosahedron within it.

G. Five Tetrahedra Exterior If the compound of five tetrahedra were made of solid interpenetrating tetrahedra, you would see only the exterior portions of their edges. You would also see lines where the planes intersect. A model of this can be seen as 60 irregular concave pentagons, as shown in the figure. Make a concave 5-gon in the yellow plane consisting of b_1–b_1–g_1–g_2–g_2, with a 60-degree angle between the b_1s, a 60-degree angle between the g_2s, and a concave angle of almost 90 degrees between the b_1 and the g_1. Make a model of the five tetrahedra using this polygon, in either left-hand or right-hand form. (Because the faces lie in the planes of an imaginary icosahedron at its center, this is a *stellation* of the icosahedron. See Unit 22.)



Compound of five tetrahedra

H. Five Small Tetrahedra Make a compound of five g_1 tetrahedra. You can try to weave the g_1s directly, following the pattern of the g_2 model, or first locate the vertices of a b_0 dodecahedron as scaffolding. (This is nicely accomplished with an outward "V" of two y_1s above every edge of the dodecahedron.) As a further challenge, build the compound of five gb_1 tetrahedra. It is just barely possible and holds itself together very snugly.

UNIT 12

The Archimedean Solids

Students examine one Archimedean solid, the icosidodecahedron, in detail. Then they find and construct the others and study and compare their symmetry.

Goals

- To become familiar with the Archimedean solids
- To review vertex notation and ideas about symmetry and counting
- To gain more insight into the relationships between polyhedra

Prerequisites

Students should have completed Unit 5. Units 8 (Tessellations) and 10 (Descartes' Theorem) are helpful.

Notes

Another way to organize the lesson is to start with 12.2, Archimedean Solids and Notation; omit the Challenge, or use it as an Exploration; and end with 12.1, The Icosidodecahedron, and 12.3, Archimedean Solids in the Zome System.

12.1 The Icosidodecahedron

First, students study the icosidodecahedron. You may have a discussion of all the ways this polyhedron is related to the dodecahedron and the icosahedron. Students then search for and construct the Archimedean solids. These can be divided among the class to be constructed and examined in detail by different groups. The class can produce a chart listing the properties of each solid (*V, E, F,* number of each type of face, and symmetry).

12.2 Archimedean Solids and Notation

For building in this unit, you will need to duplicate the regular polygons on pages 89–91 on heavy paper or, better, on light cardboard. One approach is to duplicate them on card stock. More conveniently, you can use the pre-cut polyhedra available from Key Curriculum Press. Those have tabs, and faces can be connected to each other with the help of rubber bands.

Teacher Notes

Share with your students some history of these polyhedra: Archimedes was an ancient Greek mathematician, physicist, and inventor who is known among many other things for his work on the value of π and the volume of the sphere. Around 250 B.C. he wrote a book that described the Archimedean polyhedra. Archimedes' book is lost to us. Various artists in the 1400s and 1500s discovered individual examples of these polyhedra, but not the mathematical idea of searching for a complete set. Finally, the astronomer and mathematician Johannes Kepler rediscovered the idea of looking systematically for all polyhedra with regular faces and identical vertices. Working logically, he found the full set and published them in 1609.

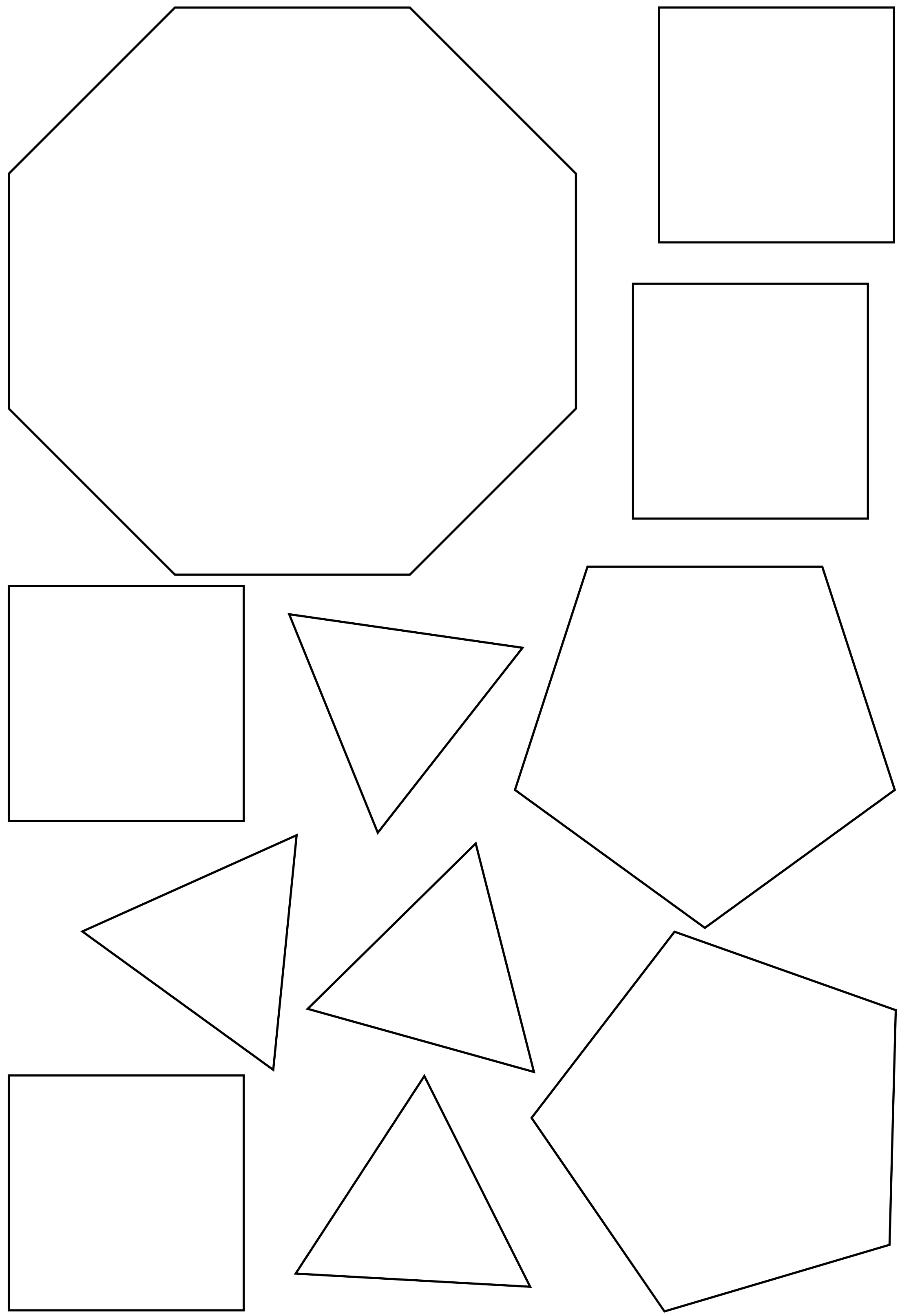
If your students have trouble finding all the Archimedean solids or if you are pressed for time, you may speed things up by handing out a copy of Kepler's drawings on page 92 or projecting them on the overhead. This can support student building, or even substitute for it if you want to limit your materials to just the Zome System.

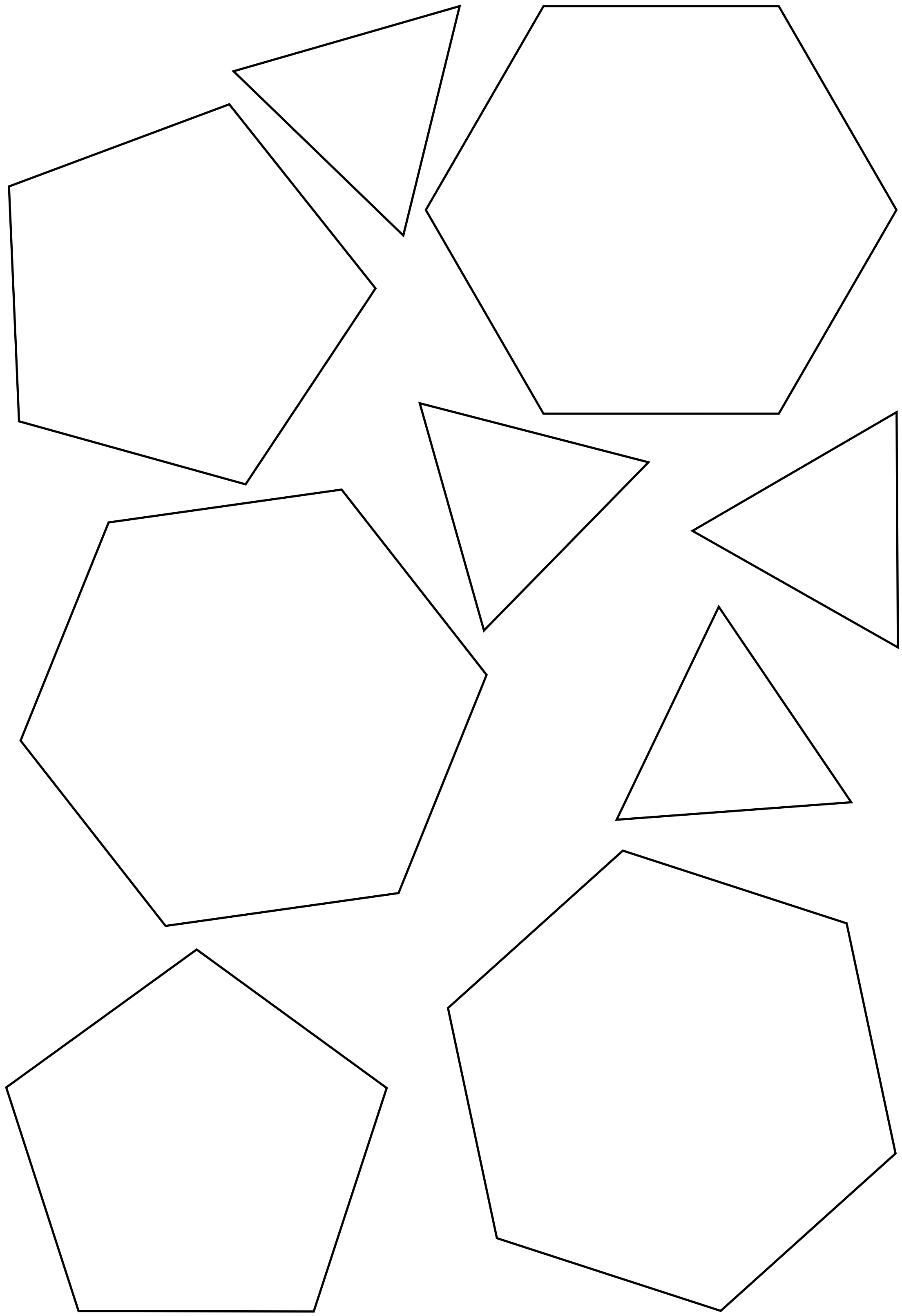
12.3 Archimedean Solids in the Zome System

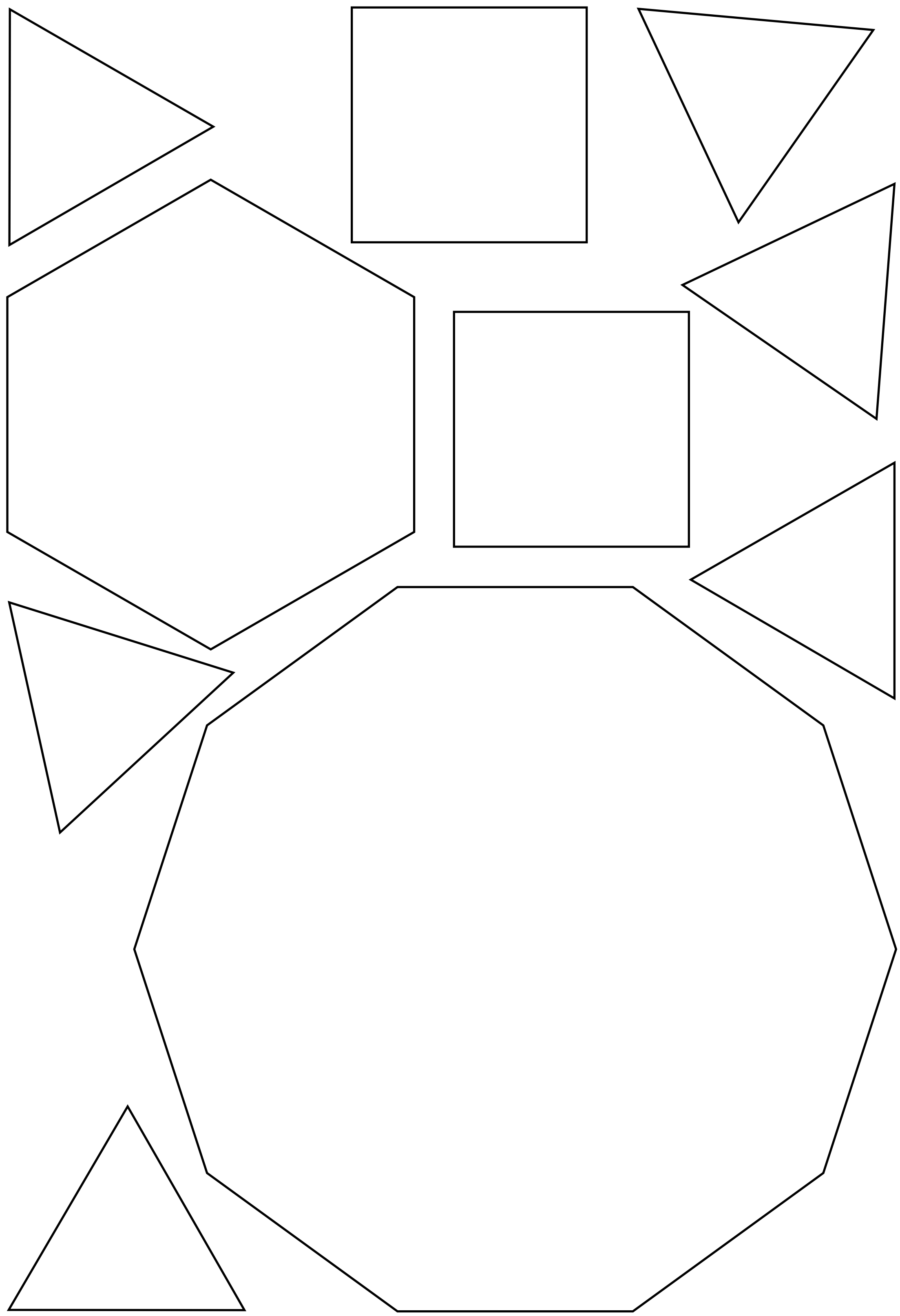
Make sure that within each class your students build as many solids as possible. This makes for the best possible discussion of tetrahedral versus icosahedral versus octahedral symmetry and of the relationships among polyhedra.

Many students will find it easier in general to build the largest polygon on the list and figure out how the other polygons join it. Other students will use the hints, which often involve building a larger solid and truncating it. Remind students about truncation: Chopping off a k-fold corner of a polyhedron reveals a k-gon face. After chopping all the corners of the polyhedron, in each plane where there was an n-gon face, you now find a $2n$-gon face. You can usually apply truncation to a triple-scale model of a polyhedron. Truncation to the edge midpoints is applied to a double-scale model.

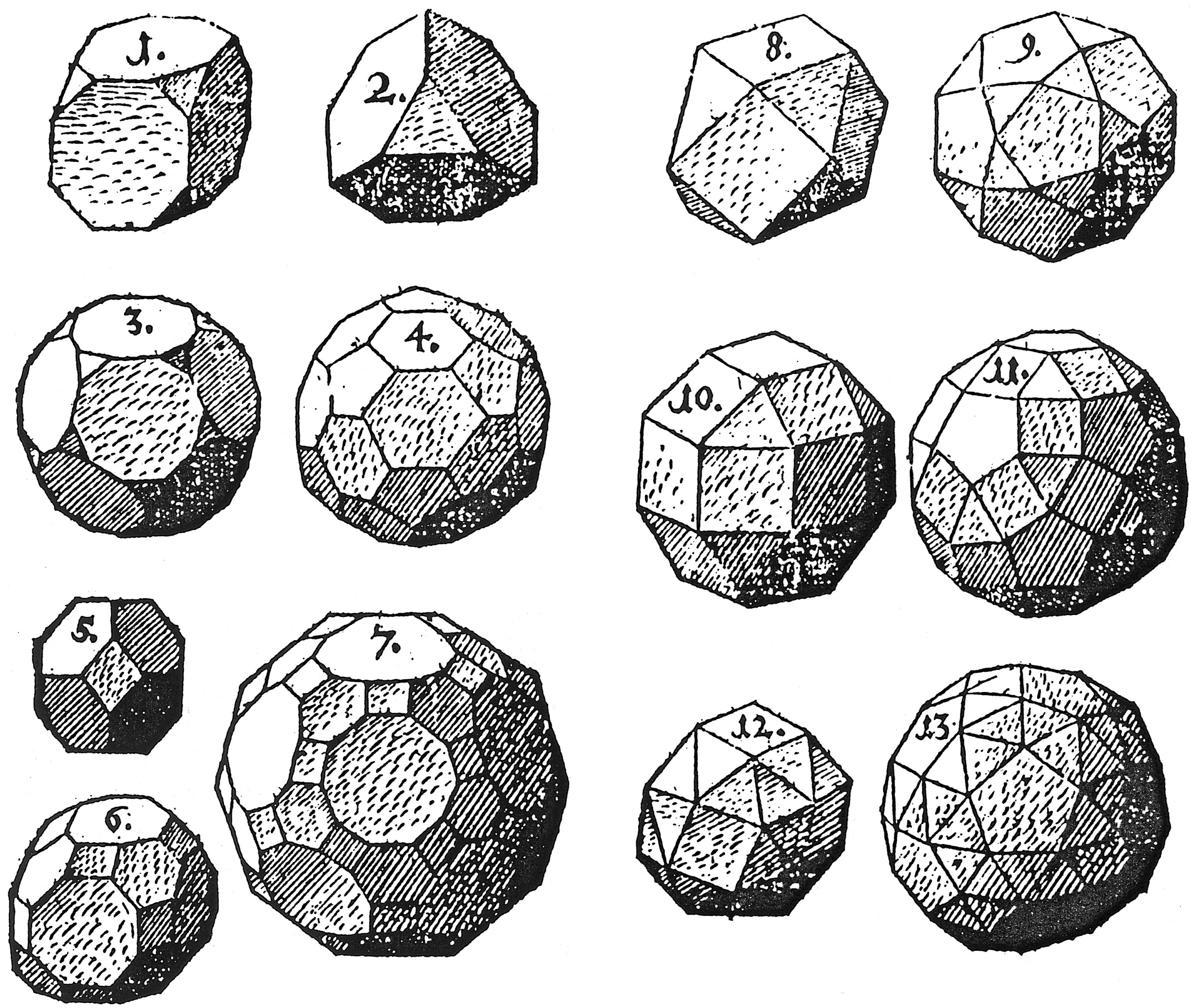
Many of these solids require green struts. Explorations E and F also require green struts.







Kepler's Archimedean Solids



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12.1 The Icosidodecahedron

Challenge

Find and build as many Zome polyhedra as you can whose faces are equilateral triangles and regular pentagons. Do not use any other polygons for faces, and do not build a figure all of whose faces are the same polygon. (Hint: There are six convex solutions. Five can be obtained by removing struts cleverly from an icosahedron. The sixth is the icosidodecahedron, one of the Archimedean polyhedra.)

1. The icosidodecahedron is the most regular and interesting of the convex polyhedra whose faces are exclusively equilateral triangles and regular pentagons. If you have not already made an icosidodecahedron, make one now, using the fact that each vertex is surrounded by a pentagon, a triangle, another pentagon, and another triangle; or use a blue starburst (30 b_2 or b_3 struts in a single ball) as scaffolding.

Q1 Count the number of triangles and pentagons in the icosidodecahedron.

Q2 Why do you suppose it is called an icosidodecahedron?

Q3 Count the edges. (Find several strategies, and make sure the results agree.)

Q4 Count the vertices. (Find several strategies, and make sure the results agree.)

Q5 Find all the symmetry elements for the icosidodecahedron.

Q6 How does the symmetry of the icosidodecahedron compare with the symmetry of the icosahedron and the dodecahedron?

12.2 Archimedean Solids and Notation

Challenge

What symmetric convex polyhedra are composed of more than one type of regular polygon?

Archimedean solids are solids that are symmetric, convex, composed of more than one type of regular polygon, and meet the condition that the vertices are *equivalent*—that is, each vertex joins the same faces in the same way.

Q1 Explain why the following polyhedra are *not* Archimedean solids.

a. a pyramid with a square base and four faces that are equilateral triangles

b. the Platonic solids

c. the zomeball (thinking of it as a polyhedron composed of triangles, rectangles, and pentagons)

Prisms and antiprisms with regular faces also meet the conditions of Archimedean polyhedra, but because there are infinitely many of them, it is traditional to count them as separate groups.

It is convenient to have a notation for what meets at each vertex. The icosidodecahedron is denoted by (3, 5, 3, 5), meaning a triangle, a pentagon, another triangle, and another pentagon meet at each vertex. [Since the pattern goes around the vertex in a cycle, this would be the same as (5, 3, 5, 3); but, by convention, the smallest number is listed first.]

1. Using paper regular polygons and tape, work with your neighbors to construct as many Archimedean solids as you can.

Even though they are not classified as Archimedean solids, regular-polygon–based prisms and antiprisms can be written in this notation.

Q2 What is the symbol for a regular *n*-gon prism?

Q3 What is the symbol for a regular *n*-gon antiprism?

Q4 Explain why (6, 6, 6) and (8, 8, 8) cannot denote an Archimedean solid.

Q5 Make a list of possible Archimedean solids using the notation. Start from polyhedra you built, but also proceed logically, using what you know about the sum of the angles at the vertex.

2. Working with your classmates, use paper regular polygons and tape to build the remaining Archimedean solids. You should end up with 13 different models.

12.3 Archimedean Solids in the Zome System

Challenge

With your neighbors, build the first 11 models. Starting from the notation, arrange those polygons around a vertex; then continue to make every vertex identical. Polyhedra marked with ⋆ require green struts.

a. (3, 6, 6) ⋆*truncated tetrahedron*

b. (3, 8, 8) ⋆*truncated cube*

c. (4, 6, 6) ⋆*truncated octahedron*

d. (5, 6, 6) *truncated icosahedron*

e. (3, 10, 10) *truncated dodecahedron*

f. (3, 4, 3, 4) ⋆*cuboctahedron*

g. (3, 5, 3, 5) *icosidodecahedron*

h. (3, 4, 4, 4) ⋆*rhombicuboctahedron*

i. (3, 4, 5, 4) *rhombicosidodecahedron*

j. (4, 6, 8) ⋆*truncated cuboctahedron*

k. (4, 6, 10) *truncated icosidodecahedron*

l. (3, 3, 3, 3, 4) *snub cube*

m. (3, 3, 3, 3, 5) *snub dodecahedron*

If you could not determine how to arrange the necessary polygons around a vertex, here are recipes for the first 11:

a. (3, 6, 6) *truncated tetrahedron* Truncate a triple-scale regular tetrahedron. (If you don't have green struts, truncate the triangular pyramid with $3b_1$ base and $3r_1$ sloping edges.)

b. (3, 8, 8) *truncated cube* Truncating a $3b_1$ cube gives irregular 8-gons, which can be adjusted into regular 8-gons using gb_1s. Alternatively, you can just assemble six octagons like the squares of a cube. Start with two perpendicular regular octagons, and join them by having them share a blue strut.

c. (4, 6, 6) *truncated octahedron* Truncate a triple-scale regular octahedron. (If you don't have green struts, make the triangular antiprism with $3b_1$ bases and $3r_1$ zigzag, and truncate it.)

d. (5, 6, 6) *truncated icosahedron* Make a triple-scale icosahedron and truncate it.

e. (3, 10, 10) *truncated dodecahedron* Make a dodecahedron with $b_1 + b_2 + b_1$ edges and truncate it, to get regular b_2 10-gons.

f. (3, 4, 3, 4) *cuboctahedron* Make a double-scale cube or regular octahedron, and truncate to the edge midpoints. (If you have no green struts, make this red-blue approximation to the cuboctahedron: Wherever the cuboctahedron has a square, the approximation will have a rectangle with two b_1 edges and two r_1 edges—or scale up, using b_2 and r_2. To get started, remove four edges from your approximate octahedron, leaving a nearly square pyramid. Place it rectangle-down on the table. Then build a red and blue tetrahedron out from each of the four triangles. Next, remove the center ball and the eight struts connected to it, leaving just the base square and four red-and-blue right angles in vertical planes. Complete those right angles into rectangles and add a top "square.")

g. (3, 5, 3, 5) *icosidodecahedron* Truncate to the edge midpoints a double-scale dodecahedron or icosahedron.

h. (3, 4, 4, 4) *rhombicuboctahedron* Start with a regular octagonal prism with square sides, that is, b_1 + gb_1 octagons as bases assembled into a prism using b_1s. Stand it vertically on a blue square face, and there will be two vertical blue square faces halfway up. Using those blue squares as a pair of opposite sides, make another regular 8-gon prism, perpendicular to the first one. Finally, eight more greens complete it as a polyhedron of squares and triangles. Notice the three mutually perpendicular octagonal prisms.

i. (3, 4, 5, 4) *rhombicosidodecahedron* Make a large version of the zomeball, but with squares instead of the rectangular holes.

j. (4, 6, 8) *truncated cuboctahedron* Make six regular octagons using b_1s and gb_1s. With one octagon flat on the table, hold another vertically so that they have two blue edges parallel and near each other. Turn one octagon 90 degrees if necessary to be sure the zomeballs are parallel. Using two gb_1s, join those two blue edges into a square that is in a plane 45 degrees from the planes of the two octagons. Similarly, attach three more octagons to the other three blue edges of the bottom octagon, by creating three more 45-degree squares. With additional green struts, create more squares joining the four sides to each other and a top octagon.

k. (4, 6, 10) *truncated icosidodecahedron* Start with a regular 10-gon and build a square on the outside of every other edge. On the remaining five edges, complete the hexagons. Make five more 10-gons, one touching each square, and continue.

Q1 Summarize the Archimedean solids. Make a table that displays

a. the number of faces, edges, and vertices

b. its symmetry elements

c. the names of the Platonic solids with the same symmetry

Explorations 12

A. Tessellations and Polyhedra Reconsider the tessellations of Unit 8. Which are analogous to Platonic solids in having only one kind of regular face? Consider what happens when these are truncated. The results are analogous to certain Archimedean polyhedra. Describe this with the $\{p, q\}$ or $(n, m, \ldots)$ notation. You can build triple-scale and double-scale tessellations to truncate.

B. Archimedean Duals Each of the Archimedean solids has more than one type of face but only a single kind of vertex. Therefore, their duals have more than one type of vertex but only a single type of face. Starting with a double-scale icosidodecahedron or cuboctahedron, you can construct a compound of it and its dual. For the other Archimedean solids, the Zome System does not have the necessary angles to make the duals, but you can sketch them.

C. In a Tetrahedron It is a curious fact that each of the Archimedean solids can be placed in a regular tetrahedron so that four faces lie in the planes of the tetrahedron's faces. For each Archimedean solid, find four faces that are in the planes of a regular tetrahedron. (For the ones related to the cube, there are two ways to choose the four faces. For the ones related to the icosahedron, there are five ways.)

D. Expanding Polyhedra A paper polyhedron can be expanded into a more complex polyhedron by the following process: First, cut the faces apart on the edges but keep them floating in space at the same relative angles; second, slide them apart from each other and the center slightly to open up space between the edges; third, insert a rectangle to connect the two parts of each cut edge; and fourth, insert an *n*-gon face that connects the remaining edges of the rectangles at each of the originally *n*-way vertices. Starting with a cube, you insert 12 rectangles (one for each cube edge) and 8 triangles (one for each three-way vertex of the cube). By adjusting the expansion, you make these rectangles become squares and the result is the rhombicuboctahedron. Try this operation on each of the Platonic solids and the icosidodecahedron.

E. Isomers Some Archimedean solids can be disassembled and rearranged into different arrangements of the same faces. For example, the cuboctahedron and the icosidodecahedron have regular polygons for "equators." When you cut on an equator, twisting the two halves and putting them back together gives a different arrangement of the same faces with some triangles adjacent to triangles. When you use a Zome model, the halves will not reconnect after the twist (since the symmetry has been destroyed), but you can hold the halves together or make a paper model to get the idea. Find two other Archimedean solids that can be cut into two parts with a regular polygon slice and analogously reconfigured.

F. **Snubs** The snub Archimedean solids cannot be constructed with the Zome System, but a form with the same arrangement of vertices and edges as the snub dodecahedron can be made. Starting with a rhombicosidodecahedron, use a green strut to add one diagonal to each square in such a way that, when you are done, every vertex has five edges. (For the first square, you can choose either diagonal; then, after that, all the other choices are determined. The two choices give the left-hand and right-hand forms.) This form is topologically equivalent to the snub dodecahedron, but its triangles are not equilateral. Visually, it should be clear that if the struts were stretchy, rotating each pentagon a small amount would allow "adjusting" the shape to make every face regular.

G. **Quasi-Regular Solids** Two of the Archimedean solids have special properties, which put them in a class of polyhedra called *quasi-regular.* In these polyhedra, every edge is equivalent, the dihedral angles (the three-dimensional angles between the planes of two faces) are equal, there are regular polygons for equators, and the solid can be formed as the interior of a compound of a Platonic solid and its dual. Which two are quasi-regular? Which compounds of a Platonic solid with its dual lead to them?

H. **Operations on the Platonic Solids** Recall that $\{p, q\}$ symbolizes the Platonic solid composed of p-gons meeting q at each vertex. Verify that the Archimedean solids can be derived from Platonic solids by applying these operations:

truncation:	$\{p, q\} \rightarrow (q, 2p, 2p)$
truncate to midpoints:	$\{p, q\} \rightarrow (p, q, p, q)$
expand (see Exploration D):	$\{p, q\} \rightarrow (p, 4, q, 4)$
truncate to midpoints, then truncate:	$\{p, q\} \rightarrow (4, 2p, 2q)$
snub:	$\{p, q\} \rightarrow (p, 3, q, 3, 3)$

What happens when the snub operation is applied to the tetrahedron? Also try these operations on the regular tessellations: $\{4, 4\}$, $\{3, 6\}$, and $\{6, 3\}$.

UNIT 13

Zome Lengths and Angles

In Unit 1 students figured out a number of zomeball angles, and in Unit 7 they calculated the relationships between Zome struts of a given color. In this unit, they will first establish the relationship between the lengths of different-colored struts and then find all the remaining zomeball angles.

Goals

- To find the relationship between Zome lengths of different colors
- To calculate the remaining zomeball angles (some green angles are left for the Explorations)

Prerequisites

Along with the angle basics in Unit 1 and the golden ratio in Unit 7, students need to know the Pythagorean theorem and right triangle trigonometry. They also need to have some facility with algebraic manipulation.

Notes

13.1 Lengths

It is easy for students to lose perspective when being led step by step through many calculations. To keep them alert, emphasize three big ideas:

- The use of the Pythagorean theorem
- Simplifying expressions with the help of $\tau^2 = 1 + \tau$
- The existence of straightforward trigonometric relationships between blue and red struts and between blue and yellow struts

The first two questions review what students learned in Unit 7. The final question is a good way to verify that your students know how to use the information they just uncovered about the relationship between the lengths. A hint for this problem is to remember which starburst scaffolding led to a dodecahedron in Unit 2.

Summary table of strut lengths in terms of τ if $b_1 = 1$:

Strut	b_1	b_2	b_3	y_1	y_2	y_3	r_1	r_2	r_3
Length	1	τ	τ^2	$\frac{\sqrt{3}}{2}$	$\tau\frac{\sqrt{3}}{2}$	$\tau^2\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\tau}}{2}$	$\tau\frac{\sqrt{2+\tau}}{2}$	$\tau^2\frac{\sqrt{2+\tau}}{2}$
	1	τ	τ^2	$\sin 60°$	$\tau \sin 60°$	$\tau^2 \sin 60°$	$\sin 72°$	$\tau \sin 72°$	$\tau^2 \sin 72°$

Teacher Notes

13.2 Angles

In comparison with the previous activity, this one is quite straightforward, especially since two of the triangles were already seen in 13.1.

It may be a good idea for students to keep sketches of the zomeball holes in the neighborhood of the two sides of each given angle, so as to be able to recognize these angles later. Another technique to facilitate the work on these problems is to hold the polyhedra in such a way that the angle being explored is in a horizontal plane. Then the polar holes in the zomeballs will indicate which plane the angle is in.

13.3 Dihedral Angles

You may explain formally: Given planes *A* and *B,* which intersect in a line *L,* construct a plane *P* perpendicular to *L. P* intersects *A* and *B* in lines *a* and *b.* The dihedral angle between planes *A* and *B* is defined to be the angle between lines *a* and *b.* In the context of a polyhedron, there are two conventions for measuring dihedral angles: Some authors choose the angle interior to the two planes, as seen at the edge, giving 60 degrees as the dihedral angles between the square sides of the right equilateral 3-gon prism. Others measure dihedral angles as the angles between the face normals. Rays pointing out of the 3-gon prism's sides are separated by 120 degrees. The two methods yield supplementary angles, so it is easy to convert between them. In this activity, students will use the first method, but the face normals are often a useful intermediary.

The answers to questions in this activity are dependent on correct answers to questions in Activities 13.1 and 13.2.

13.1 Lengths

Challenge

Using the length of b_1 as 1 unit, find r_1 and y_1. (Hint: Use the Pythagorean theorem.)

Q1 Review the basics of Zome System lengths. If b_1 = 1 unit, what is the length of

a. b_2?

b. b_3?

Q2 **a.** τ is known as the ___________ ratio.

b. Express τ exactly with an expression involving a square root.

c. Express τ as a decimal approximation.

Q3 What are the lengths of g_1 and g_2?

1. Build a b_1 cube, and find a way to connect a pair of opposite vertices with yellow struts.

Q4 Use the Pythagorean theorem in the model you just built to find the length of the cube's diagonal. (Hint: For a first step, insert the diagonal of one face if you have a green strut; otherwise, imagine it.)

Q5 Use that information to find the length of all the yellow struts.

2. Make a right triangle with legs b_1 and b_3 and a yellow hypotenuse.

Q6 Use the Pythagorean theorem in the triangle you made to find the length of y_2.

You probably found different expressions for y_2 in Questions 5 and 6. In order to prove that they are equivalent, you will have to manipulate algebraic expressions containing τ.

Q7 Remember that a fundamental and very useful property of τ is that $\tau^2 = \tau + 1$. Verify this property.

This property can be used to reduce any term τ^n into a form that is a linear combination of 1 and τ. In fact, in any expression involving integer multiples of powers of τ, it is possible to have τ appearing only to the first power, in a form $x + y\tau$, where x and y are integers. For example:

$$\tau^3 = \tau(\tau^2) = \tau(1 + \tau) = \tau + \tau^2 = \tau + (1 + \tau) = 1 + 2\tau$$

Q8 Fill in the rest of this table. (You can verify any row with a calculator, using 1.61803 for τ.)102

Power of τ	Constant	Coefficient of τ	Expression
τ^{-2}			
τ^{-1}			
τ^{0}	1	0	1
τ^{1}	0	1	τ
τ^{2}	1	1	$1 + \tau$
τ^{3}	1	2	$1 + 2\tau$
τ^{4}			
τ^{5}			
τ^{6}			

Q9 What patterns do you see in the table?

Q10 Use what you learned to show that the two expressions you have for y_2 are equivalent. (Hint: $\sqrt{1 + \tau}$ simplifies nicely.)

3. Make a right triangle with legs b_1 and b_2.

Q11 Use the Pythagorean theorem to find the length of r_1.

Like y_1, you can use trigonometry to express r_1 without involving τ.

4. Make a b_1–b_2–b_2 isosceles triangle.

Q12 Sketch the triangle, labeling all sides and angles. Draw the altitude between the equal sides, and use the figure to find the sine of 72 degrees in terms of τ.

Q13 Use algebra to show that $r_1 = b_1 \sin 72$.

Q14 Summary: Express y_1 and r_1 in terms of b_1 and the sines of appropriate angles.

5. Make a b_1 equilateral triangle and a b_1 regular pentagon. The necessary angles do not exist in the zomeball to actually build the following, but you can see that
   a. the altitude of the triangle has length y_1.
   b. a perpendicular dropped from a vertex of the pentagon to the extension of an adjacent side has length r_1.

   Q15 Check that this confirms the results you stated in Question 13.

   Q16 What is the distance between opposite vertices of a regular dodecahedron with an edge 10 meters long?

13.2 Angles

Challenge

With or without the help of a protractor, estimate any angles between a pair of struts that you have not already figured out, not including green struts.

When discussing angles, it is handy to have terminology for describing the plane the angle is in. The three types of planes will be named by the color of the strut that fits in the holes at the poles when the plane is the equator. For example, with pentagonal holes as north and south poles, you can place ten blue struts in *the red plane.*

Q1 Review: Check that you know how to find the angle between any two blue struts. List the possible angles. (Reminder: First figure out which plane the two struts are in; then see how many struts in a flat starburst would divide that plane into equal angles.)

Q2 There are two Zome right angles other than the blue-blue right angle. Describe them.

1. The blue plane contains some important angles. Place four blue, four red, and four yellow struts into one zomeball's blue plane.

This plane also includes the 45-degree angles between blue and green struts, but leave the greens out for now so that the reds can fit. For the small angle between a blue and a yellow strut, use α (Greek letter *alpha*). For the small angle between a blue and a red strut, use β (Greek letter *beta*).

2. Construct a right triangle that contains α. Construct another right triangle that contains β.

Q3 Use trigonometry to find a formula for these angles. Then use a calculator to find approximate numerical values for these angles.

You can now determine any angle in the blue plane by appropriately adding or subtracting right angles, α, and β.

Q4 What are the angles in a red rhombus?

Q5 What are the angles of a skinny yellow rhombus (that is, one in the blue plane)?

Q6 What is the angle between an adjacent red and yellow strut (in the blue plane)?

Examine a zomeball, and you should see that you now know how to determine the angle between almost any pair of struts. Not counting the green struts, there are just two angles not already covered.

3. Build an isosceles triangle made of two y_1s and a b_1.

Use γ (Greek letter *gamma*) to name the blue-yellow angle in that triangle. Note that there is no hole at the pole of the zomeballs; the polar direction is one possible direction of the green struts, so this triangle lies in the green plane.

Q7 Characterize γ using trigonometry and give a numerical value for it.

Q8 What is the angle between the two yellow struts in the isosceles triangle made of two y_1s and a b_1?

You now have seen every possible angle between two Zome struts (except the greens), so you should be able to answer all the questions below. Express your answers in terms of α, β, and γ.

Q9 What is the angle between any edge of a cube and the cube's long diagonal?

Q10 What is the central angle between two adjacent vertices of a dodecahedron? (The *central angle* means as seen from the center of the object.)

Q11 What is the central angle between two adjacent vertices of an icosahedron?

Q12 What is the central angle between two adjacent vertices of a cube?

Q13 What is the central angle between two adjacent vertices of a regular tetrahedron? Even if you don't have green struts, you can make a structure that places a ball at each vertex of a regular tetrahedron, with all four connected to a central ball.

4. As there is no hole to use as its pole, the green plane is less apparent. To find some of its features, do the following exercises (if you have green struts). Make a $2y_1$ fat rhombus. Then, in its plane, discover how to build
   - **a.** two types of isosceles triangles
   - **b.** a right triangle
   - **c.** a square-root-of-two rectangle

5. Still in the same plane, make a square.

6. To check that the constructions in 4 and 5 are indeed in the green plane, verify that you can place a long green strut vertically in a zomeball of each figure you have made.

13.3 Dihedral Angles

Challenge

Adjacent faces of a cube meet at a *dihedral angle* of 90 degrees. What dihedral angles occur between adjacent face planes in the other Platonic solids?

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1. Make a blue and yellow equilateral 3-gon prism. Remove two blue struts so that only two rectangles remain. The result is like a partly open door, where one yellow strut is the line of the hinge.

   Q1 What is the dihedral angle between the two rectangles?

Dihedral angles depend only on the planes, not the edges. Even if the rectangles were transformed into parallelograms, as long as they stayed in these two planes, the dihedral angle between them would be unchanged. Or imagine a circle in the plane of each rectangle; the circles' dihedral angle is still defined as the angle between the planes.

2. Make a parallelogram in the plane halfway between the two rectangles. One edge should be the yellow strut common to the two rectangles. Remove one of the rectangles.

   Q2 What is the dihedral angle between the parallelogram and the rectangle?

3. Add a second rectangle opposite the first, again sharing the yellow strut. Add a second parallelogram opposite the first parallelogram, again sharing the yellow strut. This model makes clearer that the planes extend on both sides of their line of intersection.

To see the dihedral angle, sight along the yellow hinge line in this model. Notice the three-fold and mirror symmetries of the holes in the ball. Most of the holes lie in planes that are separated by 30 or 60 degrees.

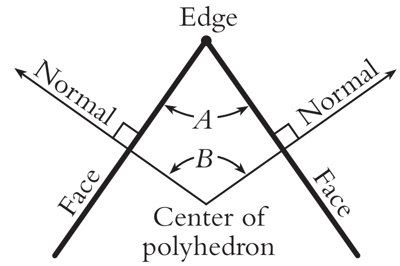
   Q3 Can you see how there are two possible values that one might reasonably call the dihedral angle between the planes? What are they? (Hint: Both are between 0 and 180 degrees.)

A *face normal* is an imaginary ray perpendicular to a face.

4. Build two blue squares. Insert a blue strut into any ball, perpendicular to the plane of each square, representing its face normal. Hold the blue squares vertically, next to each other, with the perpendicular struts back-to-back facing away from each other. Holding the balls where the face normals emerge, one in each hand, slowly turn them so that the squares separate like the opening of a door. Keep one pair of vertical edges—representing the hinge line—as close to each other as the balls allow.

Q4 As the planes separate, the normals get closer. Find and explain the numerical relationship between the dihedral angle and the angle between the normals.

5. This figure with face normals shows the dihedral angles of the regular tetrahedron. Make a b_1 or b_2 cube. Add six green struts to make a tetrahedron that uses four of the cube's vertices. From the other four vertices, add yellow struts that connect to a ball at the center of the cube.



Cross section of faces

The yellow struts are the face normals for the tetrahedron. One yellow normal passes through the center of each face. The angle between the two yellow struts was determined in Activity 13.2 to be 2γ, approximately 109.5 degrees.

Q5 What is the dihedral angle of the tetrahedron?

6. Make (part of) an icosahedron. Include enough of a starburst to locate a ball at its center. Include two struts that represent the face normals of two adjacent faces.

Q6 What is the dihedral angle of the icosahedron?

Q7 Use the same approach to find the dihedral angle of the dodecahedron.

7. Using green struts, make a regular octahedron. Attach a regular tetrahedron to one face. (This is like part of the stella octangula.)

Q8 What relationship do you see between the dihedral angle of the tetrahedron and the octahedron? Using this observation, what is the dihedral angle of the octahedron?

In the Platonic solids, all the edges are equivalent. But in most polyhedra, there are several types of edges and several types of dihedral angles to be identified.

8. Make a red rhombus. Insert four yellow struts into its four vertices at an angle that is as close as possible to perpendicular to the plane of the rhombus. Connect the ends of the new struts with a second rhombus. The result is a parallelepiped.

Q9 What are the four different dihedral angles of this parallelepiped? You can see them from the symmetry of the ball's holes or by constructing blue face normals.

Explorations 13

A. Right Triangle? Can you construct a right triangle with sides b_3, $4b_1$, and $5r_1$? Analyze it.

B. A Formula for τ^n Let F_n be the nth Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. State the general formula for τ^k. Prove it by induction.

C. Angle β The angle β also can be expressed very simply as $\beta = (1/2)\arctan 2$. Show this using the trigonometric double-angle formula

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

(The zomeball does not permit angles to make a right triangle with an angle 2β.)

D. Angles Between Lines In a plane, any two lines either intersect or are parallel. In three-space there is a third possibility. Consider a cube, call one edge *A,* and find four other edges with the property that they neither intersect nor are parallel to *A.* Call one edge *B.* Although *A* and *B* do not meet, you can still define an angle between them. Construct a line *C* parallel to *B* through any point of *A.* The angle between *A* and *C* is taken as the angle between *A* and *B* (or, equivalently, 180 minus that). With the Zome System, it is easy to construct parallel lines, because every ball is always parallel to the others. Make a list of every possible angle found between two edges of a dodecahedron. Make a list of every possible angle found between two edges of an icosahedron. Coincidence?

E. Some Green Angles There are a great many angles that involve green struts. Some are easily found from isosceles triangles. Build three isosceles triangles, each with g_2s as sides, using b_1, b_2, and b_3 as the three bases. (For one of them, two greens will want to be in the same hole, so be creative.) Are these triangles in any of the planes you have seen before? You can express their apex angles as trigonometric functions.

F. Another Green Angle One significant green-blue angle shows up in the icosahedron-in-the-octahedron construction of Unit 11. Rebuild that model and see two different equilateral triangles—one green and one blue—in the yellow plane. You can measure with a protractor that one is rotated relative to the other almost 40 degrees. How can you find the exact angle between them? (Hint: Recall that if you know the three sides of a triangle, you can determine its angles using the law of cosines: $C^2 = A^2 + B^2 - 2AB\cos\theta$.)

UNIT 14

Zonohedra

Students start by building and analyzing the properties of polyhedra in which every face is a rhombus. Students then look at other zonohedra with all-parallelogram faces and discover numerical patterns governing the number of faces and zones they have. Finally, they learn about the *star* of a zonohedron, the set of all directions of its edges, and look at zonohedra whose faces are not limited to quadrilaterals.

Goals

- To build a range of zonohedra
- To analyze zonohedra properties

Prerequisites

Students need to know the Zome basics from Units 1–5. For Activity 14.3, it is helpful if students are familiar with Archimedean solids.

Notes

14.1 Rhombic Zonohedra

The rhombic dodecahedron was built in Explorations 11, and the rhombic triacontahedron in Explorations 2.

The idea presented in this section can be used to create polar zonohedra with an axis of symmetry of any order, such as seven-fold, and so on, but the Zome System has the angles only for three-fold, four-fold, and five-fold examples. Other models can be created in different media. The angle between the axis and the initial set of struts can be chosen arbitrarily, and the process used here to create zonohedra determines the rest of the polyhedron. Exploration B suggests ways to study rhombic polyhedra further.

14.2 Zones

Zones are what make zonohedra interesting. The Zome System was named by contracting *zone-dome* into one word. The architectural possibilities of zonohedra (as implied by Questions 7–10 and Model 3) make them an attractive alternative to geodesic domes.

14.3 Stars

Here students look at zonohedra whose faces are not all quadrilaterals.

14.1 Rhombic Zonohedra

Challenge

Find several polyhedra in which every face is a parallelogram.

Any polyhedron composed entirely of parallelograms is one kind of zonohedron. You will see other zonohedra, and a definition of the term, later.

The term *rhombohedron* might have been used to refer to any polyhedron composed of rhombi, but conventionally it refers only to *hexahedra* (polyhedra with six faces) in which the faces are identical rhombi. The cube is a special case of the rhombohedron.

1. There is only one shape of red rhombus. With it, make two shapes of rhombohedron (each with 12 red edges) and notice the difference between them.

Each of the two models you made is a nonright prism on a rhombic base. They are usually called an *acute* rhombohedron and an *obtuse* rhombohedron according to which kind of angle contacts the three-fold axis.

2. Make two shapes of yellow rhombus. Use the fatter one to make two different kinds of rhombohedron.

A *rhombic hexahedron* has six faces that are each rhombi, although they need not all be congruent. The rhombohedron is the special case of rhombic hexahedron in which all the faces are congruent.

3. In blue or yellow, construct a rhombic hexahedron that is not a rhombohedron. Build a different one from your neighbor's rhombic hexahedra.

    Q1 Why is it impossible to make a red rhombic hexahedron that is not a rhombohedron?

Notice that each of the rhombohedra has a three-fold symmetry axis. This suggests another method of constructing them, starting with the idea of symmetry.

4. Another way to build rhombohedra is to start with a short yellow strut in a zomeball. This strut serves only as a three-fold axis and is not part of the final structure. Place three struts of any color but all the same size into other holes of the zomeball, except in the equator perpendicular to the axis, so that the construction is symmetric about the three-fold axis. Each pair starts a rhombus. Finish each of the three rhombi by adding two parallel edges. Three more edges meeting at a vertex opposite the original vertex complete the rhombohedron.

The method generalizes to create rhombic polyhedra with arbitrary n-fold symmetry. These are called *polar zonohedra* because they start with a pole and are built up in layers toward the other pole.

5. Make a five-fold polar zonohedron using this method. Start with a red axis of symmetry. Remember that all faces should be rhombi. There will be four layers, with five rhombi each, before the figure closes at the opposite pole.

Q2 How many faces does your five-fold zonohedron have? How many different-shaped faces?

There are exactly four polyhedra with more than six faces in which all the faces are identical rhombi:

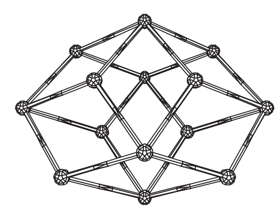
- The five-fold polar zonohedron consisting of 20 red rhombi
- The *rhombic dodecahedron* consisting of 12 yellow rhombi
- The *rhombic dodecahedron of the second kind* consisting of 12 red rhombi
- The *rhombic triacontahedron* consisting of 30 red rhombi

6. Build a rhombic dodecahedron by one of these two methods:

a. Place a yellow pyramid on each face of a cube, and then remove the edges of the cube.

b. Build a yellow four-fold polar zonohedron around a blue axis.

Two acute red rhombohedra and two obtuse red rhombohedra fit together to form a compact convex solid called the *rhombic dodecahedron of the second kind* (to distinguish it from the more symmetrical yellow rhombic dodecahedron), as shown in the figure.


Rhombic dodecahedron of the second kind

7. Construct the rhombic dodecahedron of the second kind.

8. Build a rhombic triacontahedron, which has 30 red rhombus faces, by one of these two methods:

a. Build a b_2 or b_3 icosahedron and erect a shallow red triangular pyramid on each face. Then remove the icosahedron.

b. Build a dodecahedron and erect a shallow red pentagonal pyramid on each face. Then remove the dodecahedron.

All the polyhedra you have seen in this activity are zonohedra. A *zonohedron* is a convex polyhedron such that every face has an even number of sides and opposite sides are parallel.

14.2 Zones

Challenge

Choose a zonohedron you built in the previous activity, and go "around the world" in the following way: Put your finger on an edge, and jump to the parallel edge of the same face. Repeat, until you are back where you started. For the zonohedron you picked, how long is the shortest trip around the world? How long is the longest?

1. Take either your yellow or red rhombic dodecahedron and choose any edge. Notice that you can find a set of six edges parallel to your chosen edge (including itself). A set of parallel edges is called a *zone* of edges, and the belt of faces they join is a *zone* of faces.

   Q1 Count the zones on your zonohedra.

The number of zones is the number of different edge directions.

2. Choose any two zones and see that they cross twice, at two opposite, parallel faces.

In general, if there are n zones, each zone crosses the other $n - 1$ zones at two places.

Q2 Explain why, on a given parallelogram face, a zone crosses only one other zone.

Q3 In a zonohedron whose faces are all parallelograms, if there are n zones, how many faces does each zone have? Explain.

For a zonohedron made of parallelograms, the number of zones determines the total number of faces.

Q4 Complete this chart:

Zonohedra	Number of zones (edge directions)	Number of faces
rhombohedra, parallelepipeds	3	6
rhombic dodecahedra		
five-fold polar zonohedra		
rhombic triacontahedron		

Q5 Suggest a formula that determines the number of faces given the number of zones.

Q6 Using the observation that every pair of zones crosses twice (at two opposite faces), explain why your formula works.

Q7 Imagine removing a zone of six parallel edges from one of your rhombic dodecahedra so that you have two disconnected caps, one from each side of the zone. If you slide the two caps together (and reduce the duplicated edges to single edges), they would join into what kind of polyhedron?

Q8 Take the rhombic triacontahedron and choose any edge. Notice it is part of a zone of ten edges. If you removed the zone and joined the two caps, what would be the result?

Q9 If you removed one zone from the red five-fold polar zonohedron and joined the two caps, what would be the result?

Zone removal is a special, extreme case of stretching and contracting zones.

3. Take any rhombic hexahedron (for example, any rhombohedron). See how the twelve edges come in three zones of four parallel edges each. (The cap on each side of a zone is just one rhombus.) Replace the four edges of one zone with either longer or shorter struts, changing four of the rhombi into parallelograms but keeping all the edge angles unchanged. You can change a second set of four parallel edges to the third length and create a polyhedron in which all six faces are non-rhombic parallelograms.

A hexahedron in which all six faces are parallelograms is called a *parallelepiped.* By lengthening or contracting edges, you can stretch any parallelepiped into a rhombic hexahedron, without changing its angles. However, with Zome lengths this may not be possible.

Any of the zones of any zonohedron can be stretched or contracted to create a topologically equivalent zonohedron with all the same face angles and edge directions. This process just moves the two caps closer together or farther apart.

In Questions 7–9, you thought about removing a zone from a zonohedron to obtain a smaller zonohedron. That can be reversed, to build up a zonohedron, zone by zone, from smaller zonohedra.

4. Take a 6-sided zonohedron, divide it into two halves, and add three vertices and edges to make each half three parallelograms. Keeping these halves parallel to each other, insert a zone of six new parallel edges connecting the halves. The result will be a 12-sided zonohedron.

Q10 What is it about zonohedra that makes them so easy to stretch and shrink? Can you think of a practical use for this characteristic?

14.3 Stars

Challenge

Create a zonohedron some of whose faces have more than four sides.

The process of removing zones can be reversed to create larger zonohedra from smaller ones, adding one zone at a time. To specify the set of directions for building a new zonohedron, use a *star.*

The *star of a zonohedron* is the set of its edge directions. Recall how every zomeball is parallel to all the others it connects with. You can use this fact to transfer all the directions of the edges of a zonohedron to a single ball. For each edge direction (and edge length) of a zonohedron, place two parallel struts in opposite holes of a parallel zomeball. For example, given the b_1 cube, its star has six b_1 struts placed into perpendicularly arranged holes. If you hold the star near each of the cube's vertices in turn (without rotating it), you see that the arrangement at each vertex can be found in the star.

1. Create a star for the yellow or red rhombic dodecahedron. They have four edge directions, so there will be eight struts in your star. Hold it near each vertex to verify that all the edge configurations are present.

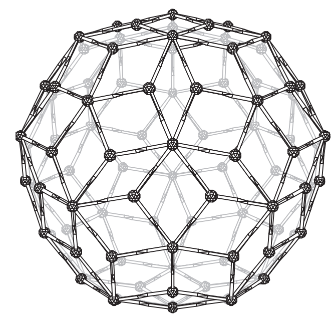
Given any star, you can create a zonohedron from it.

2. Create a random three-direction star, using blue, red, and yellow struts. It will have six struts, in three opposing pairs. Be sure they are not all in one plane. Opposite struts should be the same length, to indicate the length of the edges in that direction. Now make the parallelepiped that is the zonohedron based on that star. Copy the red and blue struts and their included angle onto another ball. Complete the "V" to make a red and blue parallelogram. Copy a yellow strut and its angles to any of the four balls of the parallelogram. Complete it into a parallelepiped by adding three parallel yellow struts and a second red and blue parallelogram.
3. Add a strut of any color to your star for a fourth direction. Be sure no three struts are coplanar. Also add the strut directly opposite your new one. This makes a star with four directions. Maintaining parallelism of the first three directions, copy the new direction to the parallelepiped. You should be able to place seven struts altogether that all point in this direction. Expand your parallelepiped to an irregular rhombic dodecahedron by connecting the ends of these seven struts. (The four internal struts dissect it into four parallelepipeds. They show how you could have chosen the directions from the star in any order and still ended up with the same result.)

Q1 What is the zonohedron based on the red starburst (the six-direction star with all twelve red struts)?

The zonohedron based on the yellow starburst is called the *rhombic enneacontahedron.* Its faces are the two shapes of yellow rhombus.

Q2 How many faces are in each zone of the rhombic enneacontahedron? How many faces does it have altogether? How many edges? If you have time, build part of it.


Rhombic enneacontahedron

4. Construct a hexagon using two b_1s, two b_2s, and two b_3s, in opposite equal pairs. All of its angles should be 120 degrees. Expand it into a right prism using a yellow strut of any length. This is a zonohedron with four zones. The rectangles form one zone. The other three zones each consist of two rectangles and the two hexagons; these three zones all cross at the two hexagons.

5. Construct the star of this zonohedron and notice that all three of its blue directions are in a single plane.

In general, zonohedra may have faces with more than four sides, such as hexagons. Like parallelograms, these faces have edges in opposite pairs of equal length. The edges of these faces correspond to groups of three or more coplanar directions. In a face with k sides, $k/2$ zones cross simultaneously. Each zone enters the k-gon face on one edge and comes out the opposite edge.

Notice that the formulas for the numbers of faces in zonohedra with all four-sided faces do not apply if some faces are hexagons.

Q3 Three of the Archimedean solids are zonohedra. Which ones? (Hint: In which ones does every face have an even number of sides?)

Q4 How many zones are in each of the three?

6. Construct the star for each of the three.

Explorations 14

A. Rhombic Diagonals The rhombic dodecahedron and the rhombic triacontahedron have special relationships with the Platonic solids.

a. If you put the short diagonal into every face of a rhombic dodecahedron, what do you get? What if you do the long (green) diagonal instead?

b. If you put the short diagonal into every face of a rhombic triacontahedron, what do you get? What if you do the long diagonal instead?

c. In a double-scale model of either the rhombic dodecahedron or the rhombic triacontahedron, insert all long and short diagonals.

B. Rhombohedra Angles Explore rhombohedra by making 12 paper copies of a given rhombus and taping them together to create the acute and obtuse rhombohedra. You will find that if the acute angle is too small, you can make only one of the two models. What is the cutoff angle? Why?

C. Zome Rhombohedra How many shapes of rhombohedra can be made with all b_1 struts? You will find that certain combinations of angles that can be made individually cannot be assembled at a single vertex, so some polyhedra that might be made in paper cannot be made with the Zome System, even though their individual faces are Zome-constructible.

D. Flat Zonohedra The polar zonohedron recipe in the first activity of this unit specifies that you cannot start with a strut in the equator. What happens if you do?

E. Green Zonohedra Green struts can be used in any of the zonohedra constructions as long as the edges are kept parallel. For the five-fold polar zonohedron, there are five ways to put the first green strut into a red hole when starting the first edges of the rhombi, but two of these are mirror images of others, so there are only three distinct angles. Each leads to a five-fold polar zonohedron. (There is also the mathematical possibility of starting with five green ribs all in the hole of the axis. This gives a very long zonohedron, but it is not Zome-constructible.) How many green three-fold polar zonohedra (rhombohedra) are there?

F. Six-Fold and Ten-Fold Zonohedra Build a six-fold almost-polar zonohedron, starting with three red and three yellow ribs arranged like a half-open umbrella. Generalize this to a ten-fold, almost-polar zonohedra in blue and red or in blue and yellow.

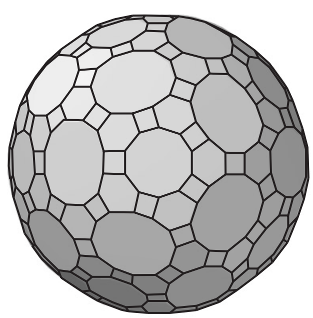
G. **Skew Polygons** There is a close relationship between regular skew polygons and polar zonohedra, which can be used for finding and counting all the skew polygons. There is a regular skew 2*n*-gon around the "equator" of any *n*-fold polar zonohedron when *n* is odd. Look for it on sample polar zonohedra. Conversely, every regular skew 2*n*-gon can be used as a "seed" to construct a polar zonohedron. Build some regular skew polygons, and see if you can build the polar zonohedron that corresponds to each one. How does this one-to-one matching help you find and count regular skew polygons?

H. **Zonohedra Symmetry** Describe the symmetry elements of each of the zonohedra in this unit.

I. **Negative Edges** What happens if you shrink the edge length of one or more zones to a negative length?

J. **31-Zone Zonohedron** If you put struts into all 62 holes of a zomeball, you have a 31-direction star that generates a 31-zone, 242-sided zonohedron. Ignoring green struts, it is the largest Zome-constructible zonohedron and the largest strictly convex polyhedron that can be built with the Zome System using a single strut per edge. Its largest faces have 12 sides. If you are careful, you can build it in short struts.

Connection



This 31-zone, 242-sided zonohedron is the largest convex polyhedron you can build with red, blue, and yellow struts. Zonohedra are of interest architecturally because a wide range of structures may be formed with a small inventory of parts.

Steve Baer invented the Zome System around 1970 for architectural applications.

UNIT 15

Area

In the first activity, students calculate the surface areas of various Zome figures. In the second, they review the idea that the ratio of the areas of similar figures is the square of the scaling factor.

Goals

- To become familiar with various techniques for determining area
- To review the lengths of struts and the angles between them

Prerequisites

Units 1, 7, and 13 are crucial, as those are the ones in which strut lengths and angles are calculated, and that is the foundation of any work on Zome area. Students should also be good with a calculator, and they need an understanding of right triangle trigonometry for many of the problems.

Notes

15.1 Calculating Area

This activity is a straightforward application of basic ideas about area. The Challenge provides a chance to jump right into it and find ways to get the area of triangles and parallelograms, which are the most likely candidates for smallest area.

If you want students to work on this activity at home, have them sketch all the needed figures in class, including length of struts and angle measurements.

See the book *Journey Through Genius: The Great Theorems of Mathematics,* by William Dunham, for a nice presentation of Heron's proof.

15.2 Scaling Area

The Challenge is an opportunity to review the golden-ratio relationship between struts of the same color. You can also follow it up with a calculation of the rectangle areas in terms of $\mathbf{b_1}$, which would also review the length relationships between colors.

One way to find the area of a triangle scaled with ratio of similarity k (if k is a natural number) is to break it up into rows of triangles of the original size. The k rows have 1, 3, 5, . . . , $2k - 1$ triangles, respectively. So the scaled area is the original area times the sum of the first k odd numbers, which is k^2.

15.1 Calculating Area

Challenge

Find the Zome polygon with the smallest possible area. Do not use green struts.

Calculation of area is an important practical application of geometry; surveying land areas was one of the historical roots of geometry. In fact, the word *geometry* comes from the Greek "to measure the earth."

Our unit of area will be b_1^2. All Zome areas can be expressed in the form $c\,b_1^2$, where c is purely numeric, that is, c contains no strut lengths. Think of the b_1^2 at the end of each area as a kind of unit, analogous to meters^2.

As you know, the area of a triangle is equal to half the base times the height.

1. Make a triangle using two b_1s and a b_2.

 Q1 What is its area?

An alternate method of finding the area of a triangle was discovered and proved by Heron almost 2000 years ago. *Heron's formula* is often useful when the three sides, x, y, and z, are known. Define s to be the semiperimeter, equal to $(x + y + z)/2$. Heron's formula is Area $= \sqrt{s(s - x)(s - y)(s - z)}$.

 Q2 Verify that Heron's formula gives the same result as in Question 1.

2. Make a triangle using two b_2s and a b_1.

 Q3 Find its area using both methods.

 Q4 What is the area of a b_1 pentagon? (Hint: Divide it into triangles.)

 Q5 What is the area of a b_1 decagon?

 Q6 What is the area of an r_1 rhombus?

 Q7 Find a general formula for the area of a rhombus that has diagonals x and y.

 Q8 You can make two different y_1 rhombi. What are their areas?

15.2 Scaling Area

Challenge

Make three different-shaped rectangles with the same area, each using only four Zome struts.

In this activity, you will investigate what happens to the area of a figure when its dimensions are multiplied by *k*. To start with, consider a b_2-by-y_2 rectangle.

Q1 If you multiply the length by 3,

- **a.** is the resulting figure similar to the original?
- **b.** what happens to the area?

Q2 If you multiply the width by 3,

- **a.** is the resulting figure similar to the original?
- **b.** what happens to the area?

Q3 If you multiply both the length and the width by 3,

- **a.** is the resulting figure similar to the original?
- **b.** what happens to the area?

Q4 How is the ratio of areas related to the scaling factor?

Q5 How is the area of a b_3-by-y_3 rectangle related to the area of the b_2-by-y_2?

Algebraically, for a rectangle: $A = lw$, where l = length and w = width. After scaling with a scaling factor k: $A' = lk \cdot wk = lwk^2 = Ak^2$.

The same effect happens with any figure.

1. Make a triangle that is different from your neighbors' triangles.
2. Make a triangle with triple the dimensions of the original, and dissect it into copies of the original triangle.

Q6 How many original triangles fit into the larger one?

Q7 Make an algebraic argument like the one following Question 5, but apply it to triangles.

In general, the ratio of area is the square of the scaling factor.

Q8 What is the area of a triangle made from two b_2s and a b_3?

Q9 What is the surface area of an icosahedron with edge length 3 meters?

Q10 If the distance between opposite edges of a dodecahedron is 1 inch, what is its surface area? (You can check to see you are not way off by comparing the answer to the area of a sphere 1 inch in diameter, if you recall that a sphere has surface area $4\pi r^2$.)

Explorations 15

A. Regular *n*-gons There is a natural method of dividing a regular *n*-gon into $2n$ congruent right triangles that meet at its center. You cannot make a Zome model of these triangles, because they have a right angle and an angle of $^{180}/_{n}$ in the same plane. Give a formula for the area of a regular *n*-gon with edge length *e.* (Draw this for some value of *n,* then use trigonometry.)

B. Rhombic Dissection See how many different ways you can find to dissect a regular Zome 10-gon into rhombi. Notice you always end up with ten rhombi, five with a 36-degree angle and five with a 72-degree angle! Why?

C. Midscribed Polyhedra A Platonic solid can be *inscribed* in a sphere, which means its vertices lie on the sphere. It can also be *circumscribed* around a sphere, which means its face centers lie on a sphere and the faces are tangent to the sphere. A third choice is that it can be *midscribed* to a sphere, which means its edge midpoints lie on the sphere, so its edges are tangent to the sphere. The inscribed polyhedron lies inside the sphere and has less surface area than the sphere. The circumscribed polyhedron lies outside the sphere and has more surface area than the sphere. But a midscribed polyhedron is partly inside and partly outside the sphere. Which parts are inside? Which do you think is closer to the area of a sphere: a midscribed icosahedron or a midscribed dodecahedron?

UNIT 16

Space Structures

Extending the idea of covering a plane with a tessellation of polygons, students find polyhedra that will fill space. They look first at polyhedra that will fill space by translation, then they find solids that fill space with translation and rotation. Space filling is then extended to packing spheres, in this case zomeballs.

Goal

- To build and understand structures that can fill space

Prerequisites

Unit 8 or experience with tessellations will give students background with the two-dimensional equivalent of space filling. Students will need to be able to build a stella octangula (Unit 3), a truncated octahedron and the cuboctahedron (Unit 12), and the rhombic dodecahedron (Unit 14). They need to know that the dihedral angle of the regular octahedron is 2γ, approximately 109.5 degrees (Unit 13).

Notes

The extensive building in Activities 16.1 and 16.3 takes time but is valuable for developing three-dimensional visualization. Green struts are used for some constructions, but alternate directions for a red-blue approximation are also given.

16.1 Filling Space

As students build and visualize the five groups of polyhedra that can fill space by translation alone, they will be determining the dihedral angles at the edges. You might ask students to determine the dihedral angles when they build polyhedra that fill space with translation and rotation in Model 7.

The fact that only five convex solids can pack in this way was proved a century ago by the Russian crystallographer E. S. Federov, who discovered zonohedra while studying the possible atomic arrangements in crystals.

16.2 Packing Spheres

Note that the balls and struts are not connected to each other in this activity.

16.3 The FCC Lattice

This activity requires green struts. You may choose to use red-blue approximations for the octahedra and tetrahedra.

This activity reviews several key ideas that appear in the first two activities. The FCC (face-centered cubic) pattern of balls and the octet truss arrangement of edges are closely related but are not equivalent. If equilateral, the octet truss vertices correspond to the FCC lattice of balls. However, the balls could be connected in other ways than the edges of the octet truss, and the octet truss might be stretched or compressed to have rhombi instead of squares.

16.1 Filling Space

Challenge

Construct some interesting, infinite, space-filling structures—a three-dimensional equivalent to two-dimensional tessellations.

Space structures are structures that can be replicated to fill all space, analogous in three dimensions to the tessellations. For example, cubes can be stacked to fill space. So can hexagonal prisms—just think of the hexagon tessellation, thicken it into prisms, and stack layers on top of each other. The thickening need not be done at a 90-degree angle—the prisms need not be right prisms.

Another space-filling structure is the triangular prism made by thickening a tessellation of triangles. But there is a subtle difference between the triangles and the hexagons or cubes. When you pack together triangular prisms, some are turned 180 degrees relative to others, but when you stack cubes or hexagonal prisms, they are always parallel to each other. They are all translated copies of each other, without rotation.

There are only five convex shapes that can be used to fill space by translation alone, without rotation. The cube and hexagonal prism are two. The rhombic dodecahedron is a third.

1. Build a rhombic dodecahedron around a cube by adding a yellow pyramid to each cube face.
2. Build a cube with yellow struts joining its vertices to its center.

Imagine space filled with cubes, and mentally color the cubes alternately black and white, like a three-dimensional checkerboard. Every cube has six neighbors of the opposite color. Now imagine the white ones divided into six square pyramids, as in the model you just built. Each black cube is empty and surrounded by six pyramids. Imagine a Zome model of this, and remove the blue struts. This will leave a structure of rhombic dodecahedra filling space.

3. Build a portion of this space structure. (If you use cubes as scaffolding, remove them so that you can see the yellow space structure you created.)

Q1 How many rhombic dodecahedra surround each edge in this space structure?

Q2 What size are the dihedral angles at each edge of the rhombic dodecahedron?

4. To build an *elongated rhombic dodecahedron,* a fourth polyhedron that will fill space by translation, take a rhombic dodecahedron and locate a square equator that divides it in half. Carefully separate the two halves and add four zomeballs to the bare strut ends. Now reconnect the halves in their original orientation, but separated from each other, by adding four blue struts. You are expanding four rhombi into four hexagons.

   Q3 Explain how the elongated rhombic dodecahedron can fill space. What surrounds an edge? Is there only one type of edge?

   Q4 What are the dihedral angles of the elongated rhombic dodecahedron?

5. To build the fifth space-filling polyhedra, start with a green truncated octahedron or a red-blue approximation.

6. Expand your model by connecting it face-to-congruent-face to other truncated octahedra, continuing until you have a sense of how this could be continued indefinitely in all directions, without leaving any gaps between the polyhedra.

   Q5 What surrounds each edge in the structure of truncated octahedra? Is there only one type of edge?

   Q6 What are the dihedral angles at each edge? (Hint: Think of the truncation process to see that the hexagon-hexagon edge has the same dihedral angle [2γ] as the original octahedron. Then, knowing what fits around an edge in the space structure, you can solve for the dihedral angle of the hexagon-square edge.)

These five solids that fill space by translation are all zonohedra. They fit together in part because opposite faces are identical and parallel, and so provide a match to the next unit in that direction. Stretching the zones or changing the angles (for example, skewing a cube into a parallelepiped or the rhombic dodecahedron into the rhombic dodecahedron of the second kind) does not affect their ability to pack space, so each of the five solids really represents a family of shapes with different lengths and angles.

If rotation is allowed when polyhedra fill space, there are countless possibilities for space-filling structures.

7. Choose one of these space-structure possibilities and build it:

a. a square pyramid that can fill space

b. a rhombic pyramid that can fill space

c. an irregular tetrahedron that can fill space

d. an irregular octahedron of eight triangles that can fill space

e. a nonconvex polyhedron that can fill space

8. Build a space structure consisting of regular octahedra and tetrahedra (or their red-blue approximations). Each polyhedron should be surrounded on all sides by copies of the other.

Q7 What surrounds any interior edge of the structure?

Q8 What are the dihedral angles at each edge?

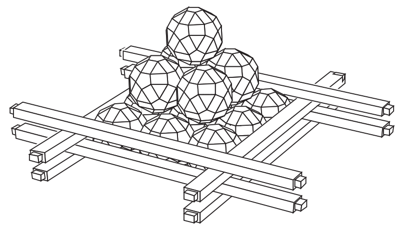
9. Build a *cuboctahedron* (3, 4, 3, 4) or its red-blue approximation. Expand it into a space structure consisting of cuboctahedra and regular octahedra. (Hint: Since the octahedra have no square faces, the cuboctahedra must connect square-face-to-square-face.)

16.2 Packing Spheres

Challenge

Find some ways to pack zomeballs into a box. Can you fit more of them into the box by stacking them directly on top of each other or by placing balls in the next layer in between balls in the previous layer?

Pyramids of cannonballs form regular sphere packings. You may have noticed them with either a square base or a triangular base. Both can be re-created with stacks of zomeballs. To hold the zomeballs from rolling away, create a frame of struts like a log cabin.



Log cabin frame of struts with pyramid of zomeballs

1. To create a square frame, lay two b_2 struts on a flat surface, parallel to each other, about two inches apart. Rest two more across those, to outline a square. Then lay two more over the first pair and another two over the second pair. Place nine zomeballs in the frame, as a 3-by-3 square, and tighten the frame to eliminate any free space. Four more balls can be added for a second layer and one on top as the third layer, to complete a square pyramid.

   Q1 If this structure were expanded infinitely, how many neighboring balls would each ball touch?

   Q2 In a horizontal slice of the structure, touching balls are arranged in a square array. What patterns of touching balls can be found in other plane slices? (Hint: Look at planes parallel to the triangular faces of the pyramid you built, and look for vertical planes.)

2. To examine a triangular pyramid of zomeballs, use six b_3s stacked like a triangular log cabin to make a frame to hold a 4–3–2–1 triangle of zomeballs. Place a 3–2–1 triangle above that, then a 2–1 triangle, and finally a single ball.

   Q3 If this structure were replicated infinitely, how many neighboring balls would each ball touch?

   Q4 What patterns of touching balls can be found in different plane slices?

   Q5 If expanded infinitely, how would these two arrangements of balls be related to each other?

When stacking balls by either pyramid method, it is always clear where the next layer should be positioned, because the sides of the pyramid are guides. However, if you first constructed two layers of infinite extent and then went to place the third layer, there would be no edges to guide its placement.

Examine one layer of balls in a square arrangement and one in a triangular arrangement. Imagine them to be of infinite extent. In the square arrangement, the balls of the second layer can rest in the pockets of the first layer in only one way. However, in the triangular arrangement, when you choose a pocket for the first ball, the ball blocks the adjacent three pockets, so only every other pocket is used. If you started in an adjacent pocket, the first pocket would be blocked. In either case, you can construct an infinite triangular arrangement of balls, but there are two possible positions for it. The freedom in the triangular arrangement does not affect the structure if there are only two layers. But when a third layer is added, there is a choice.

3. Make a new zomeball triangle, this time five balls on a side (in a b_3 frame). Add seven balls to the second layer—one in the center surrounded by six in a hexagon. Find two different ways to place three balls as the third layer.

The balls of the third layer may be directly above balls of the first layer, or not. When laid on a triangular plane like this, when the balls are directly above other balls three layers below, it is called *face-centered cubic packing* (FCC). When balls are directly above balls two layers below, it is called *hexagonal close packing.* In either packing, any ball is in contact with 12 other balls.

Q6 In FCC packing, these 12 balls form the vertices of what kind of polyhedron? (Hint: Look at the space structures you built in the previous activity.)

Q7 In the case of hexagonal close packing, describe the polyhedron whose vertices are formed by these 12 balls.

16.3 The FCC Lattice

Challenge

Build square and triangular cannonball pyramids, but connect neighboring balls with appropriately chosen struts.

Face-centered cubic packing can be derived from three different starting points.

1. Make a $2b_1$ square as scaffolding and the X formed by its two $2g_1$ diagonals. Expand this into a cube with five more X sides—the blue cube edges are not necessary. Finally, add twelve g_1s to make the octahedron of edges that connect the six face centers, and remove anything blue.

You have a stella octangula. Stacking these cubes together to fill space results in the FCC structure.

2. Visualize how four of these units around an imaginary cube's edge create an octahedron so that every stella octangula is surrounded by octahedra and vice versa. Build this structure only if necessary.

   Q1 How is the octahedron-tetrahedron space structure visible in the space structure of packed stella octangulas?

3. Find three perpendicular planes with squares and four directions of planes with triangles.
4. Now build the FCC structure again, but starting with square pyramids. Make a 2-by-2 portion of a square tessellation. Erect a pyramid on each square. Connect the four apexes together into a square of the dual tessellation. Visualize how this can extend in three dimensions.

   Q2 In this structure, space is divided into square pyramids and what other polyhedron? Where does the other polyhedron come from?

   Q3 Visualize multiple layers of this structure. Imagine gluing pairs of square pyramids square-to-square to form a larger polyhedron. What space structure results?

5. Again, find three perpendicular planes with squares or near-squares and four directions of planes with triangles.
6. To create the same structure a third way, start with stacked rhombohedra. Think of the tessellation of squares, slanted into 60-degree rhombi. Build a 2-by-2 unit of green rhombi and expand it into a 2-by-2-by-2 unit of rhombohedra. (If you do not have green struts, the pointy red rhombohedron gives a good approximation, with angles of 63 degrees instead of 60 degrees.)

7. Imagine the pattern continuing off to infinity in all directions, and turn it various ways to see nine planes of rhombic tessellations, in three sets of three parallel planes. Notice how any of the planes could have been chosen as the starting tessellation. In these rhombic planes, the balls form the vertices of rhombi.

8. Find other planes in which the balls suggest a tessellation of equilateral triangles. Add struts to connect the balls in these planes, outlining all the equilateral triangles.

   Q4 What polyhedra are outlined?

9. Find three planes with squares (or near-squares if you are not using green struts) and four directions of planes with triangles.

A slice or two from the structure of alternating tetrahedra and octahedra is commonly used in architecture, usually with struts all the same length. It makes a strong, lightweight framework, ideal for spanning large open spaces. Because it is based on alternating octahedra and tetrahedra, architects call it the *octet truss,* a name coined by Buckminster Fuller (1895–1983), who popularized this structure for architectural applications. Depending on which slice is used, the top and bottom surfaces might be all triangles or all squares. Look up next time you are at the mall to see if you can find the octet truss applied.

Explorations 16

A. Other Rhombohedra You can repeat the stacked rhombohedron derivation of the lattice (Activity 16.3, Exercise 6), starting with a different rhombohedron—blue or yellow—instead of the green or red one you used. (Do not use b_1s, and use the cube or the flat yellow rhombohedron only if you have green struts.)

B. Tetrahedra and Octahedra When tetrahedra and octahedra alternate over a large region of space, what is the ratio of tetrahedra to octahedra? Count three different ways based on: first, how many faces each has and what each face borders; second, how many edges each has and what each edge borders; and third, how many vertices each has and what each vertex borders.

C. A Concave Polyhedron Packing Exploration F in Unit 11 considered a concave dodecahedron, which is formed by building a cube in a dodecahedron then subtracting from the cube the six roof shapes. Thinking of that construction, find an interesting space packing that involves alternating with a well-known polyhedron.

D. Bilunabirotunda A dodecahedron can be placed so that six of its edges lie in the faces of an imaginary cube. Stacking those cubes gives a space structure of dodecahedra joined at the edges. The intervening spaces can be nicely filled with two other convex polyhedra, defined by the dodecahedra vertices. One is the cube, and the other is bounded by pentagons, squares, and triangles. Construct the other polyhedron, which is sometimes called *bilunabirotunda.*

E. Archimedean Packings Here are some other combinations of Archimedean solids that can pack together to fill space. Determine how and make a model:

a. tetrahedra and truncated tetrahedra

b. cuboctahedra and octahedra

c. tetrahedra, cubes, and rhombicuboctahedra

d. cubes, cuboctahedra, and rhombicuboctahedra

e. octahedra and truncated cubes

f. truncated tetrahedra, truncated cubes, and truncated cuboctahedra

g. truncated tetrahedra, truncated octahedra, and cuboctahedra

h. truncated rhombicuboctahedra and octagonal prisms

i. cubes, octagonal prisms, truncated cubes, and rhombicuboctahedra

F. **The Diamond Lattice** Atoms of carbon form four bonds to other carbon atoms. In a diamond, the arrangement is such that each carbon atom has four neighbors at the vertices of a regular tetrahedron surrounding it. Place four yellow struts (all the same size) into one zomeball so that the angle between any two struts is identical. Place a ball at the end of each strut. The balls are at the vertices and center of a tetrahedron. Each ball represents a carbon atom, and the struts represent its four bonds to neighboring atoms. Because the angles are equal, no two neighbors are unnecessarily close to each other. (The atoms repel each other, so arrangements with very close components are not favored.) To each of the four balls, add another three struts, to make the four struts in each ball all at equal angles. Do this in such a way that every new strut is parallel to one of the four original struts. (Even though a zomeball has twenty yellow holes, only four directions are used here.) Continue the structure until you can see repeating patterns with many skew hexagons. How is this related to other space structures you have seen?

G. **Packing Regular Dodecahedra?** Might it be possible that regular dodecahedra can be packed to fill space in some way, fitting three around an edge? (You cannot solve this by trying to build a model of three around an edge, because the Zome System does not even allow you to build two face-to-face dodecahedra.)

H. **Building Blocks** Imagine a set of wooden blocks in the shapes of regular tetrahedra and square pyramids. (Both block shapes use the same size equilateral triangle.) How could you use them to build the truncated tetrahedron, the truncated octahedron, the cuboctahedron, a parallelepiped, a large regular tetrahedron, and a large regular octahedron? Make a Zome model for each.

I. **A Hexadecahedron** Build a g_1 truncated tetrahedron. On the outside of the four triangles, build low y_1 pyramids (or use g_2s and y_2s). This 16-sided polyhedron can fill space. How? Can this idea be applied to make other space-filling solids?

J. **Alternating Dual Tessellations** The packing of space with square pyramids and tetrahedra was based on the square tessellation and its dual in alternating layers. Generalize that idea to a tessellation that is not self-dual.

UNIT

17 Volume

After an introduction to volume and scaling, students find the volumes of several polyhedra.

Goals

- To become familiar with techniques for calculation of volume
- To get more experience with scaling

Prerequisites

Students need to know the results for area derived in Unit 15. They will use their experience of building the rhombohedron and the rhombic dodecahedra (introduced in Unit 11 and Unit 14).

Notes

17.1 Prisms and Scaling

For many students, this idea about scaling volume is not easy. This activity will work better as a review of the concept than as an introduction to it.

Questions 5 and 6 follow a pattern parallel to the algebraic argument used when scaling area in Unit 15.

17.2 Pyramids and Beyond

Without the hint, the Challenge is extremely difficult. An additional hint is that it can be done with the scaffolding in the plane of the pentagon; there is no need to go into three dimensions. The solution to this problem will be essential to answer later questions, so you should give away the answer if no one finds it.

In the absence of green struts, you can create models with heavy paper for Exercise 1.

17.1 Prisms and Scaling

Challenge

Find a Zome prism with smallest possible volume, without using green struts. (Hint: It will not be a right prism.)

Use the b_1 cube as the unit of volume—any Zome volume can be expressed as $c\ b_1^3$. The volume of a prism is the product of its height and the base area. If it is not a right prism, measure the height perpendicularly to the base.

1. A *golden brick* is a right rectangular prism that uses all three sizes of blue strut. Make a golden brick. Then make a cube with the same volume.
2. A *rhombohedron* has six faces, each an identical rhombus. Make two kinds of r_1 rhombohedron (each with 12 r_1 edges) and notice that each is a nonright prism on a rhombic base.

   Q1 Treating them as nonright prisms, what is the volume of each of the two rhombohedra with r_1 edges? (Hint: To find the perpendicular height, make double-scale models.)

   Q2 What is the ratio of their volumes?

3. Join together two acute r_1 rhombohedra and two obtuse r_1 rhombohedra to form the rhombic dodecahedron of the second kind, whose faces are 12 identical r_1 rhombi.

   Q3 Find the volume of the rhombic dodecahedron of the second type.

When a solid is scaled, its surface area is multiplied by the square of the scaling factor, and its volume is multiplied by the cube of the scaling factor.

4. To see this, make a right prism with a square or rectangular base. Build one different from your neighbors' right prisms.
5. Make a new right prism, with double the width, double the height, and double the depth of your original one.

   Q4 What is the ratio of the new to the old

   a. surface area? **b.** volume?

   Q5 Write an algebraic explanation of what happens when all three dimensions of a rectangular prism are multiplied by 2

   a. to the surface area. **b.** to the volume.

   Q6 Repeat Question 5, but all three dimensions are multiplied by *k*.

   Q7 What is the volume of the r_3 rhombic dodecahedron?

   Q8 How tall a prism on a b_1 square base would it take to have the same volume as the structure in Question 7?

17.2 Pyramids and Beyond

Challenge

Construct a regular pentagon (any size) and its center. That is, connect some zomeball to the pentagon with a pattern of struts that places that zomeball in the exact center of your pentagon. (Hint: First try to make, in the same plane, a regular pentagon and a regular decagon that are concentric.)

1. If you have green struts, you can show one example of how the volume of a pyramid is one third the volume of a prism of the same base and height. Build a cube. From the same vertex, build three face diagonals and one diagonal through the center of the cube. Study the resulting model to see that you have created a dissection of the cube into three congruent square-based pyramids. The volume of each pyramid is one third of the volume of the cube and therefore one third of base times height. This result holds for all pyramids.
2. Make a pyramid on a b_1 square, with the apex above the center of the square.

   Q1 What is its volume?

The volumes of more complex polyhedra may be determined by dissection into simpler polyhedra such as prisms and pyramids.

Q2 The yellow rhombic dodecahedron can be built by adding pyramids on all the faces of a cube. What is the volume of a y_1 rhombic dodecahedron?

Q3 What is the volume of a pyramid with a b_1 triangle base and r_1s as the slanting edges? (The height of the pyramid can be found with a triple-scale model.)

Q4 What is the volume of an icosahedron with edge 1 meter? (Hint: Dissect an icosahedron into triangular pyramids.)

Q5 Describe a strategy to find the volume of a b_1 dodecahedron.

Q6 How can you find the volume of a b_1 pentagonal-based pyramid with y_2 slanting edges?

Q7 What is the volume of a b_1 dodecahedron?

Explorations 17

A. Algebraic Notation Simplifying expressions involving τ and $\sqrt{5}$ can be difficult. An early 1800s geometry text gives the following method for determining the volume of a dodecahedron from the edge length: "To 21 times the square root of 5 add 47, and divide the sum by 40: then the square root of the quotient being multiplied by 5 times the cube of the linear side will give the solidity required." It is not obvious that this is equivalent to our answers. Express this method in modern notation rather than words, then verify with a calculator that it gives the same answer as the simpler expression derived in Activity 17.2. Can you see advantages to modern notation over this textual presentation of the formula?

B. Antiprism Volume To calculate the volume of an antiprism, it is handy to know the prismatoid formula. A *prismatoid* is a generalization of the prism and the antiprism. It is any solid with two parallel polygons as bases (they need not be regular or equal) connected with a "circumference" of triangles or quadrilaterals (not necessarily rectangles or isosceles triangles) that touch both bases. The volume is $(1/6)(A_1 + 4A_2 + A_3)h$, where h is the height, A_1 and A_3 are areas of the two bases, and A_2 is the area of the cross section halfway between the planes of the bases. For example, in a pentagonal antiprism, A_1 and A_3 are pentagons and A_2 is a decagon. Notice that in a prism, $A_1 = A_2 = A_3$, so the formula reduces to A_1h as you expect. Explain why if A_3 shrinks down to a point, the formula reduces to the formula for the volume of a pyramid.

C. Dissecting the Rhombic Triacontahedron The rhombic triacontahedron can be dissected into 30 pyramids, each on a rhombic base. Summing them, find the volume of the rhombic triacontahedron. You can also dissect a rhombic triacontahedron into 20 red rhombohedra, 10 acute and 10 obtuse. Make a model showing this dissection, and verify that summing the volumes of the 20 rhombohedra gives the same rhombic triacontahedron volume you obtained by summing pyramids.

UNIT 18

Big Domes

Students will construct structures big enough for one student to sit in.

Goals

- To apply some of the Zome-construction ideas to problems of an architectural nature
- To construct zonish polyhedra

Prerequisites

Students should have completed Units 14 and 16.

Notes

This is a unit that may be best suited to math clubs or extra-credit projects, rather than to actual class time. The large, complex models require a good amount of parts, floor space, time, patience, planning, organization, and cooperation. It is very rewarding to complete constructions on a human scale. However, the Zome System was not designed as playground equipment. These fragile structures are at the very limits of what the set can do.

There is no one right way to design a structure. Students will need to make informed choices to trade off between various criteria such as size, strength, parts needed, time needed, appearance, symmetry, weight, rigidity, and economy in the use of Zome equipment.

Distribute the guidelines on page 141.

18.1 Zonish Big Domes

The big domes introduced in this section are like geodesic domes but are not always as spherical. (In a geodesic dome, all the vertices lie on the surface of an imaginary sphere and all the faces are triangular; here you will not have those constraints.)

The activity introduces the geometry behind *zonish polyhedra.* Using the Zome System, any polyhedron can be expanded to have zones in any of the Zome directions. The zonish polyhedra have zones but are not zonohedra unless you start with a zonohedron. These zonal expansion ideas can be skipped if you want to go straight to the constructions without treating the underlying geometry. They are treated more thoroughly in the article "Zonish Polyhedra" by George W. Hart.

Teacher Notes

18.2 Single-Layer Domes

Because these projects are so massive, distribute the polyhedra in this activity among different groups of students. As the students work, encourage them to look at the underlying icosahedral symmetry and other relationships with familiar polyhedra.

18.3 Two-Layer Domes

The method used in Activities 18.1 and 18.2 is natural in architectural applications, such as building a mall, because it uses many identical parts, saving on cost through mass production. However, when you are working with a fixed supply of Zome System kits, a structure using four different strut types might be built four times as large as a structure using only one type. The octet truss was covered in Activity 16.3.

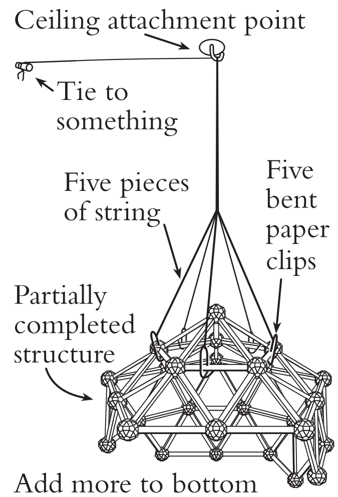
18 Guidelines for Building Big Domes

For any of the big-dome structures, read through all the instructions on the activity page to understand the entire process, make a small portion of a small-size model, visualize the finished structure, and develop a strategy.

Part of your strategy may involve counting the total number of struts you will need. Will you need to replace some b_3s with b_1–b_2s? If so, where should you place these heavier, weaker pieces?

Another aspect of the strategy is to decide whether you will build and then connect modular components or work your way around the structure, adding one strut at a time while being careful not to throw it off balance.

In some cases, it is easiest to work from the top down. Suspend the partial structure with some string and paper clips, as shown in the figure.



How to suspend your partial structure

In other cases, you can build upward, but be sure to design a base big enough to support the weight of the full building.

If you design a single-layer dome, an overall spherical form will make it strong for its weight and able to support *distributed loads*—forces that are spread out across a wide area, like wind. However, these domes have little resistance to *local* loads, which can disconnect a joint, creating a disruption that can spread to neighboring joints and lead to a complete collapse. A two-layer big dome, where the neighborhood of a vertex has a three-dimensional structure, can resist local loads better.

For greater strength, you can *triangulate* a polygon by dissecting it into triangles. In some cases, it is helpful to use diagonals even if they are not part of a full triangulation. It is also possible to reinforce a structure temporarily, as it is being built, and then carefully remove some of the scaffolding later on, to lighten the structure or to reclaim struts.

Remember that you always must push struts tightly into the zomeballs. Structures without tight connections fall apart easily, and small errors accumulate to significant proportions when making these large models.

18.1 Zonish Big Domes

Challenge

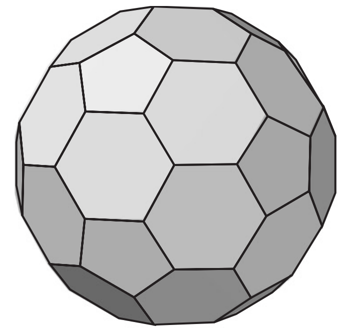
Design and build a Zome structure based on a dodecahedron that you can sit in.

Q1 Describe ways to strengthen a big dome by triangulating different Zome polygons. How many diagonals does it take to create triangles in a rhombus? a pentagon? a hexagon? a decagon? an *n*-gon?

1. Build a regular dodecahedron and rest it on a face. Look down on it along the vertical five-fold axis, and see how the exterior of its shadow is a skew 10-gon. Carefully remove the top half of the dodecahedron (a 5-gon and 5 struts) that is above this skew 10-gon. On each of the 10 balls that are part of this equator, place a red strut vertically (any size, but all the same size). Place a ball on each red strut, and connect the balls to make a second skew 10-gon, parallel to the first. Reattach the top half of the dodecahedron to this second skew 10-gon, to form a convex polyhedron with 12 pentagons and 10 parallelogram faces. This "zonish" polyhedron is expanded with one zone parallel to one five-fold axis.

2. Expand the polyhedron with a second zone. Turn your model so that it rests on another 5-gon face. The first zone of parallelograms will be tilted. Look straight down on it and see that the exterior of its shadow consists of two halves of a skew 10-gon, but where they meet there are parallelograms in vertical planes. Remove and set aside the top half, dividing it so that just the top half of each parallelogram is taken off. In each of ten balls where part of the top half was removed, insert vertically a red strut with a ball on the end. No red strut goes into the two balls that are in the middle of the parallelograms. Connect the new balls with two half skew 10-gons. Place the top back on, creating eight new parallelograms and two irregular hexagons. This dodecahedron expanded with two zones has hexagonal faces where the zones cross.

3. Continue this process step by step until all six zones are complete (one for each five-fold axis). At each step of the way, there are more parallelograms turning into hexagons and fewer edges turning into parallelograms. Once you understand the structure of the polyhedron shown in the figure, you can build it without having to add the zones one at a time.



Dodecahedron expanded with six zones

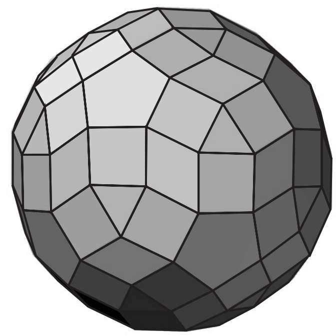
Q2 A dodecahedron expanded with six zones could also have been obtained by truncation. What is another way to describe this polyhedron? (Hint: Although it consists entirely of pentagons and hexagons, it does not have the correct number of hexagons to be a truncated icosahedron.)

18.2 Single-Layer Domes

Challenge

Expand familiar polyhedra in zonish ways to design and build a large Zome structure.

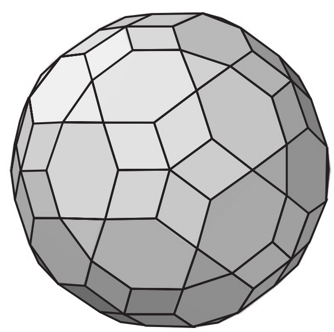
1. The polyhedron illustrated at right consists of 20 triangles, 12 pentagons, 30 red rhombi, and 60 almost-squares. It was constructed using b_3s and r_3s, so the almost-squares are red and blue rectangles. It is a zonish form derived from the icosidodecahedron by adding six red zones. Start with the pentagons and triangles as modules, then use r_3s to connect them. After the polyhedron is half built, it can be suspended, the bottom attached, and gently lowered over the creator. Or, if you are not trying to have someone sit in it, you can build it working your way around with the whole structure resting on one pentagon.



Icosidodecahedron expanded with six zones

2. The polyhedron illustrated below right, consisting of 20 equilateral triangles, 60 fat yellow rhombi, and 30 almost-regular hexagons, has icosahedral symmetry. The rhombi are grouped into twelve sets of five, positioned on the five-fold axes. The triangles are on the three-fold axes, exactly as in an icosahedron. Each hexagon has two-fold symmetry and sits on a two-fold axis. The bottom star of five rhombi can be omitted so that the whole structure rests on ten points—two points of each of five stars.

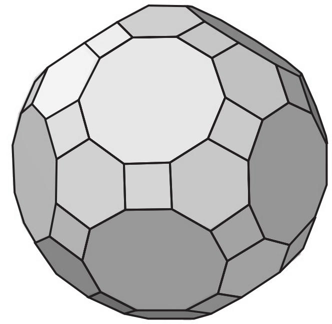
To make this structure by a modular approach, construct eleven sets of the five y_3 rhombi, then connect them with b_3s that form the triangles.



Icosahedron expanded with ten zones

This zonish polyhedron can be understood as starting from an icosahedron that was then expanded in the ten directions corresponding to the yellow, three-fold axis directions.

3. The truncated icosidodecahedron illustrated here is the largest Archimedean solid for a given edge length. In b_3 size, someone might fit inside. The model can be made almost complete, omitting one 10-gon and all the struts it contacts. This allows it to rest on ten balls distributed around the base, two from each of five 10-gons, dividing five hexagons in half, which can be braced with diagonals. This is the hardest of the three to build, because the decagons are so loose. Consider using diagonals in the lower five decagons.



Truncated icosidodecahedron

18.3 Two-Layer Domes

Challenge

Design and build a two-layer big dome.

One method for building a two-layer big dome is to take any combination of space-filling polyhedra, fill a large region of space with it, and hollow out most of it.

1. Use b_3s and r_3s to build a structure of alternating octahedra and tetrahedra. Imagine filling enough space to create a large truncated octahedron with four struts per edge. Subtract from it a truncated octahedron with three struts per edge to leave a very strong two-layer dome.

Another method is to take any polyhedron and use a slice of a space structure for its faces.

2. If you build the truncated octahedron with octet truss space structures for its faces, you would have the structure built in Model 1. Or you could use other space structures such as a honeycomb of hexagon prisms to strengthen the hexagons of the truncated octahedron.

3. To make a two-layer $2b_3$ rhombicosidodecahedron, 5 1/2 feet in diameter, start by visualizing the form. Each vertex is of the form (3, 4, 5, 4) with edges of length $2b_3$. Surrounding every triangle are three squares. Surrounding every pentagon are five squares. Surrounding every square are two triangles and two pentagons oppositely positioned.

   Make three different kinds of space-structure modules, one for each kind of face. The space-frame modules are truncated pyramids, truncated at the half-height level to reveal an *n*-gon of size b_3. The pentagonal pyramids have the form of $2b_3$ icosahedron caps. The triangular pyramids have the form of one twentieth of an icosahedron. The square pyramids have the form of one sixth of a cube.

   The bases of the space-frame modules face outward, and their edges are shared with the neighbors. The edges of the inner *n*-gons are not shared; they connect using r_2s around every triangle and y_3s around every pentagon.

   One strategy is to make the pentagonal and triangular modules separately, then build the connecting squares, which assembles the other modules. Just over half a sphere makes a good-sized dome. One pentagon can be left open as a doorway.

Explorations 18

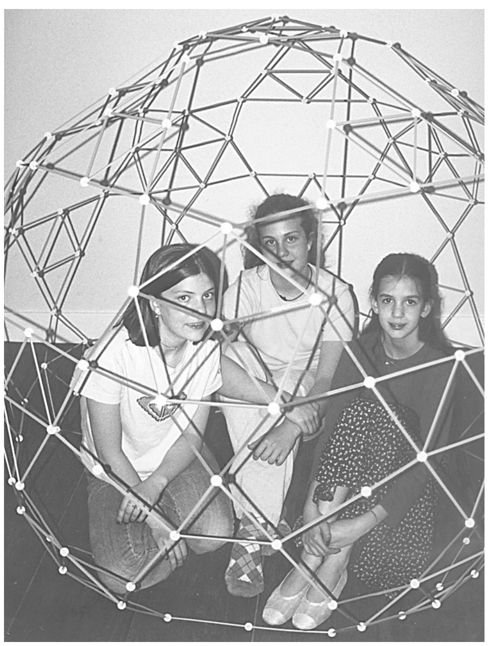
A. Pillars Another approach you might try is a flat slab of some space structure standing on pillars. The result is like a table with legs. You can also put a partial dome on pillars to raise it up. A difficulty with these approaches is that the pillars are critical weak points; bumping into one can destroy the entire structure.

B. A Zonish Dodecahedron Expand a regular dodecahedron with yellow zones in ten directions.

C. A Zonish Icosidodecahedron Expand an icosidodecahedron with yellow zones in ten directions.

D. A Zonish Cuboctahedron Expand a green cuboctahedron with four yellow three-fold axis directions.

Connection



This dome is based on a $\mathbf{2b_3}$ icosahedron expanded with ten $\mathbf{y_3}$ zones, with one $\mathbf{b_3}$ diagonal added to each rhombus.

UNIT 19

Coordinates

Students find the coordinates of the vertices of an icosahedron centered at the origin and use them in a variety of ways. They apply the same techniques to the dodecahedron.

Goals

- To find the Cartesian coordinates of the vertices of the Platonic solids and various Archimedean solids
- To use Cartesian coordinates to determine lengths in a polyhedron

Prerequisites

In addition to Units 7, 9, 11, and 13, students should know the distance formula.

Notes

In this unit, we use b_1 as the unit and write coordinates as pure numbers in terms of b_1.

19.1 Vertex Coordinates

The questions in this activity are written assuming a $2b_1$ icosahedron, but there are advantages to making a larger model: Using a $2b_2$ icosahedron makes it possible to build the dual dodecahedron right on the same model; using $2b_2$ or $2b_3$ makes it easier to reach inside the polyhedron as needed; and asking some groups to use b_2 or b_3 will make your materials go further. However, using a larger model and b_1 units leads to complex algebraic manipulations. If you use a $2b_2$ or $2b_3$ icosahedron, just use b_2 or b_3 as the unit of length so that numerically the edge is always of length 2.

Additional struts can sometimes be placed inside the polyhedra, parallel to the axes, in order to make the coordinates of the vertices more obvious.

If your students have access to programmable calculators, they can write a short program to evaluate the distance formula with specific inputs.

Teacher Notes

19.2 Point Operations

Students will use points in three-dimensional space as objects. The notation $p = (x, y, z)$ is used to designate these objects. They can be the arguments and results of mathematical operations. Students will use scaling, addition, averaging, and weighted averaging to determine the vertices of various polyhedra. The activity serves as a gentle introduction to vectors and helps students to develop an appreciation that mathematical operations may concern objects other than simple numbers.

19.1 Vertex Coordinates

Challenge

Make a **2b₁** dodecahedron and hold one vertex. Find the distances between the one vertex and all the other vertices.

In the 1600s, the French mathematician René Descartes (1596–1650) discovered that coordinate systems can be used to identify points in a plane or in three-dimensional space. Certain problems are easier to solve using coordinates than by Euclidean proof techniques. In the plane, x- and y-coordinates suffice to identify a point. In three-dimensional space, three coordinates, (x, y, z), are needed. Today, computer graphics are among the many uses of coordinates.

1. Build a $2b_1$ cube.

Q1 What are the coordinates of the vertices of a cube of edge length 2 centered at the origin and oriented parallel to the axes?

There is one vertex in each of the eight *octants* of the coordinate system, so every combination of plus and minus signs occurs once. A shortcut way to write the coordinates is $(\pm 1, \pm 1, \pm 1)$.

Q2 What would the coordinates be if the cube had edge 1 but was still centered at the origin?

Q3 Find the coordinates of the vertices of a tetrahedron centered at the origin. (Hint: Orient and scale the tetrahedron in such a way as to be able to use the coordinates of the vertices of a cube to help you.)

2. Build a $2b_1$ icosahedron. Connect a pair of edges to form a rectangle.

3. Build two more rectangles by connecting pairs of icosahedron edges, making them perpendicular to the first rectangle and to each other. This will help you see where to place your axes.

4. Build x-, y-, and z-axes so that the icosahedron is centered at the origin and each of the three rectangles is in a coordinate plane. You need to decide which axis is which. Hold the icosahedron in front of you, with one edge resting on the table going from left to right. Add some scaffolding at the bottom to keep it balanced on the edge. Call the horizontal axis going to the right the positive x-axis. Call the horizontal axis going away from you the positive y-axis. Call the third axis, going vertically up, the positive z-axis.

Q4 What are the coordinates of the vertex at the left end of the top edge in terms of b_1, b_2, or b_3?

Q5 Once a unit of length is chosen, coordinates can be written in terms of that unit. What are the coordinates of that same vertex if b_1 is the unit?

Q6 Assuming a unit of b_1 implicit in every coordinate, write the coordinates of all 12 vertices. Feel free to write $\pm$ if that helps make the list more compact.

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) can be found with the help of the Pythagorean theorem. It is given by the distance formula: The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Q7 Use the distance formula to find the length of two icosahedron edges: the edge that is bisected by the positive x-axis and the edge from $(1, 0, \tau)$ to $(0, \tau, 1)$.

5. Temporarily (just for Question 8) rotate the icosahedron 90 degrees about the x-, y-, or z-axis.

Q8 Now what are the coordinates of the vertices?

6. Build a $2b_1$ dodecahedron, look for rectangles in it, and use them to orient the dodecahedron in such a way as to have conveniently located axes. Build the axes.

Q9 What are the coordinates of the vertices of the dodecahedron?

Q10 Use the distance formula and the coordinates of the vertices of the dodecahedron to confirm the distances you found in the Challenge.

19.2 Point Operations

Challenge

Using the coordinates you found for the vertices of Platonic solids, find the coordinates of the vertices of one of the Archimedean solids.

Each triple of three numbers is a *point*. Two points can be added or a point can be scaled to determine the location of a new point.

The distance from point p to the origin is symbolized as $|p|$. To find $|p|$, use the formula for the distance between the point p and (0, 0, 0).

Q1 In a $2b_1$ cube centered at the origin, use coordinates to determine the distance from the origin to the vertices.

Q2 In a $2b_1$ dodecahedron centered at the origin, use coordinates to determine the distance from the origin to the vertices. Explain why two vertices suffice.

Scaling each coordinate scales a point. Given a number, c, and a point, $p = (x, y, z)$, multiplying the number times the point gives a resulting point, written $cp = (cx, cy, cz)$. It is calculated by multiplying each coordinate separately by c. For example, if $p = (1, 1, -1)$, then $5p = (5, 5, -5)$. Geometrically, $5p$ is a point five times as far from the origin as p but on the same line from the origin. If each of a cube's vertex points is multiplied by c, the results are the vertices of a cube scaled by c, still centered at the origin.

Q3 What are the vertex coordinates of an icosahedron that is inscribed in a unit sphere?

To find the *sum* of two points, $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$, sum coordinates separately. This gives a point

$$p_1 + p_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Geometrically, the sum gives the fourth vertex of a parallelogram that has p_1, p_2, and the origin as its other three vertices. For example, the top two vertices of our icosahedron are $(1, 0, \tau)$ and $(-1, 0, \tau)$, so their sum is $(0, 0, 2\tau)$, which is directly above the origin.

1. Build the parallelogram that represents the preceding example of summing points.

The average of two numbers is half their sum; it is halfway between them on a number line. Similarly, if p_1 and p_2 are the endpoints of a line segment, then their *average*, $(1/2)(p_1 + p_2)$, gives the midpoint of the segment.

Q4 If p_1 and p_2 are the endpoints of the top edge of the icosahedron, where is the average, $(1/2)(p_1 + p_2)$?

Q5 If $p_1 = (1, 1, -1)$ and $p_2 = (1, 1, 1)$ are two endpoints of an edge of your cube, what is the midpoint of that edge?

2. Truncating a cube to its edge midpoints gives a cuboctahedron (3, 4, 3, 4). Build one by adding **g**$_1$s to your **2b**$_1$ cube and then removing the extra **b**$_1$s.

Q6 What are the coordinates of the vertices of your cuboctahedron? Use $\pm$ to keep your answer compact.

Q7 The midpoints of the edges of an icosahedron are the vertices of an icosidodecahedron (3, 5, 3, 5). Compactly list the coordinates of the 30 vertices of an icosidodecahedron.

If $p_1, p_2, p_3, \ldots, p_n$ are the vertices of a regular n-gon, then their *average,* $(p_1 + p_2 + \cdots + p_n)/n$, is the center of the n-gon.

Q8 What is the center of the cube face with vertices (1, 1, 1), (1, 1, –1), (1, –1, 1), and (1, –1, –1)?

Q9 The centers of the faces of your cube give the vertices of a regular octahedron. What are the coordinates of the octahedron's vertices?

You will find the centers of two specific icosahedron faces. First, find three specific edges. Call the edge that is bisected by the positive x-axis the *x-edge,* and similarly define the *y-edge* and the *z-edge.*

Q10 Find the coordinates of the center of the face that has the x-edge as one edge and has one vertex on the z-edge. Then find the center of the face that touches the x-edge, the y-edge, and the z-edge.

Q11 By duality, the center-points of the icosahedron's faces give the vertices of a dodecahedron. So the two points just determined are two of the vertices of a dodecahedron. But it is scaled differently from the **2b**$_1$ dodecahedron whose vertices you found earlier. Find the scale factor that scales these coordinates to the **2b**$_1$ dodecahedron.

The *weighted average* $(1/3)p_1 + (2/3)p_2$ gives a point two thirds of the way along the segment from p_1 to p_2. It is closer to p_2 because p_2 is given more weight.

Q12 The truncated icosahedron (5, 6, 6) is derived by creating vertices at the $1/3$ and $2/3$ points along the icosahedron's edges. Find coordinates of two vertices of the truncated icosahedron, derived from the top edge of the icosahedron.

3. Build a **$3b_1$** cube and a regular tetrahedron inscribed in it. Orient it so that one of the tetrahedron's vertices is at $\left(3/2, 3/2, 3/2\right)$.

Q13 What are the coordinates of the vertices of the truncated tetrahedron?

Explorations 19

A. Rotating the Cube We saw a cube of edge 2 centered at the origin, with vertices $(\pm 1, \pm 1, \pm 1)$. If you rotate that cube 45 degrees about the z-axis, it is still a cube of edge 2 centered at the origin. What are its vertices?

B. Rotating the Icosahedron In the coordinates of the icosahedron, study how the three entries shift from one group to the next. The change $(a, b, c) \rightarrow (b, c, a) \rightarrow (c, a, b) \rightarrow (a, b, c)$ is the action of a three-fold rotation axis. Can you find the axis by studying the coordinates?

C. Fifteen Rectangles In a dodecahedron with edge length b_1, you can insert two b_3s to connect two opposite edges, making a rectangle concentric with the dodecahedron. A rectangle of this shape is called a τ^2 rectangle because the ratio of its sides is equal to τ^2. It is easy to construct three mutually perpendicular τ^2 rectangles within one dodecahedron. If you try to continue this, to make a compound of all 15 τ^2 rectangles in a dodecahedron, you run into a problem of intersecting struts. The solution is to scale up and divide the larger struts to have nodes at the intersection points. Make a dodecahedron of edge b_2 or b_3, and construct all 15 concentric τ^2 rectangles in it. Are you surprised by what forms in the center? If you then remove that inner figure, what do you have? If on the other hand you remove the original dodecahedron's edges, what do you have?

D. Three Rectangles and a Cube Build three concentric and mutually perpendicular τ^2 rectangles with short side $2b_1$. Then add a concentric cube with side $2b_2$ (and faces parallel to the rectangles). This will give you a *naked* dodecahedron: All the vertices will be there, but not all the edges. To build such a figure, start with a center ball, and use a set of three b_1 axes centered at that ball to serve as scaffolding for your rectangles. Use yellow struts to support the vertices of the cube. Can you do this in a way that does not allow the dodecahedron to be built?

E. Parallelepiped Theorem A parallelepiped is a polyhedron with six parallelograms as faces. Like a slanted, squashed cube, it has twelve edges and four diagonals through its center. Prove that the sum of the squares of the lengths of the four diagonals of any parallelepiped is equal to the sum of the squares of the lengths of the twelve edges. (Hint: Put one vertex at the origin, use nine variables, such as (x_1, y_1, z_1), to name the coordinates of the three adjacent vertices; then use the distance formula to express the theorem as an equation in the nine variables.)

F. **Computer Visualization** If you have access to software that produces three-dimensional graphic displays of given (x, y, z) points, display some of the preceding polyhedra.

G. **Truncating** To get Archimedean solids with regular faces, the one-third points of an edge are appropriate for truncating Platonic solids composed of triangles, but not squares or pentagons. Why? What weighted average is appropriate for finding the vertices of the truncated cube?

H. **Soccer Ball Dimensions** If you wanted to make a soccer ball by sewing together cloth pentagons and hexagons, you could plan its size if you imagine a truncated icosahedron inscribed in a sphere and calculate the ratio of the radius to the edge length. The earliest recorded calculation of this ratio is by the fifteenth-century painter and mathematician, Piero della Francesca (ca. 1410–1492), who studied the truncated Platonic solids. He lived before Descartes, and so used other methods, but you can find the answer using coordinates.

UNIT 20

Self-Intersecting Polyhedra

The Platonic and Archimedean solids are generalized to regular polyhedra whose faces are self-intersecting polygons and then to uniform polyhedra. These are three-dimensional versions of two-dimensional self-intersecting polygons, which are introduced first.

Goals

- To see the analogy between self-intersecting polygons and self-intersecting polyhedra
- To stimulate spatial visualization
- To build some beautiful structures

Prerequisites

Students need experience with Units 2, 3, 11, and 12.

Notes

20.1 Self-Intersecting Polygons

Uniform self-intersecting polygons are introduced; they are the two-dimensional analogs of self-intersecting polyhedra.

20.2 The Kepler-Poinsot Polyhedra

This activity is long, so plan to take extra time. The ending consists of questions that can be answered during the next class period if the models can be preserved. You might want to have one group build the second model for the great dodecahedra, one group build the third model for the great dodecahedra, one group build the small stellated dodecahedron (great icosahedron), and another build the great stellated dodecahedron. The great stellated dodecahedron is on the color insert.

You might want to share this history with your students. The German astronomer Johannes Kepler (1571–1630) recognized that pentagrams are regular polygons and studied the two stellated dodecahedra. Two hundred years later, the French mathematician Louis Poinsot (1777–1859) discovered the great dodecahedron and the great icosahedron. He recognized that they are the duals of Kepler's two stellated dodecahedra.

Teacher Notes

20.3 Uniform Polyhedra

For the Challenge, make sure students realize that when the turtle turns 120 degrees, the interior angle it traces is 60 degrees.

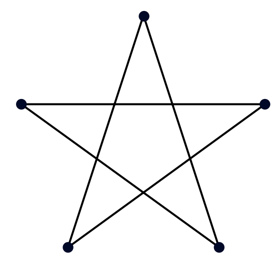
The names of the uniform polyhedra pictured here are in the index. Your students might be interested in knowing that the search for the uniform polyhedra involved many people over many years. Forty-one of the easy-to-find ones were published in a paper in 1881. The last 12 were discovered in the 1930s, but the first complete list wasn't published until a 1953 paper by H. S. M. Coxeter and others. The list wasn't proven to be complete until a computer search by John Skilling in 1975. *Polyhedron Models,* by Magnus J. Wenninger, shows paper models and templates for all of them, and his book *Dual Models* does the same for their duals.

20.1 Self-Intersecting Polygons

Challenge

Create interesting Zome designs consisting of the diagonals of regular polygons.

A *pentagram* is a symmetric five-pointed star—a polygon with five vertices and five edges. It is regular because it has equal sides, equal angles, and five-fold symmetry. It is a *self-intersecting polygon,* meaning some of its components cross through each other.



A self-intersecting polygon

1. Make a Zome pentagram.

Only the outermost zomeballs of a Zome System pentagram correspond to its actual vertices. There are also five inner balls that are not vertices; they are just places where the edges cross each other—points of intersection. You should think of each $b_2 + b_1 + b_2$ edge as a single edge.

Q1 How many edges does a pentagram have? How many vertices?

Even though it has five equal sides and five equal angles, the pentagram is not the usual regular pentagon. There is a different notation to describe this kind of figure; it will be presented after you build another example.

2. Make a b_2 decagon. Extend each of the ten edges in both directions by adding 20 b_1s. Then 10 zomeballs will connect the extended edges together.

Q2 Describe the figure you made.

3. Continue extending the edges in both directions, this time adding 20 b_2s. Again add 10 zomeballs to connect the extended edges together.

Q3 Describe the figure you made, thinking of each $b_2 + b_1 + b_2 + b_1 + b_2$ segment as a single long edge.

The result is a single regular polygon. It has ten equal edges, ten equal angles, and ten-fold symmetry. But it is a different type of 10-gon from the usual regular 10-gon. In addition to its 10 vertices, there are 20 interior points of intersection.

Notice that the vertices of this polygon are positioned exactly like the vertices of an ordinary b_3 10-gon. However, if one follows the $b_2 + b_1 + b_2 + b_1 + b_2$ edges, they connect every third vertex around the circle. This suggests a way that you could draw it on paper.

Q4 Draw the polygon you built in Exercise 3.

This polygon is a *($^{10}/_{3}$)-gon*, which means that there are ten equally spaced vertices and every third one is connected. Generally, an *($^{n}/_{m}$)-gon* is a regular polygon with *n* vertices, each of which is connected to the vertex that is *m* vertices over, around the circumference.

Q5 Draw a ($^{7}/_{2}$)-gon and a ($^{7}/_{3}$)-gon.

Q6 What is a ($^{7}/_{1}$)-gon?

Q7 What is a pentagram in this notation?

Q8 What is a ($^{10}/_{7}$)-gon?

Q9 Consider the ($^{10}/_{7}$)-gon and also the ($^{5}/_{3}$)-gon, and propose a rule about when two ($^{n}/_{m}$)-gons are identical.

Q10 What happens if you try to draw a ($^{10}/_{2}$)-gon, a ($^{10}/_{4}$)-gon, or a ($^{10}/_{5}$)-gon?

Q11 What is the interior angle at any vertex of a regular ($^{n}/_{m}$)-gon?

20.2 The Kepler-Poinsot Polyhedra

Challenge

A *pentagrammatic prism* has pentagrams (five-pointed stars) for bases. It can be constructed using five red struts to connect the vertices of two pentagrams. Each of the five rectangular sides passes through two other sides, so it is a *self-intersecting polyhedron.* Make other polyhedra in which some faces will have to cross through each other. Note: The Zome System does not have the angles for a pentagrammatic antiprism.

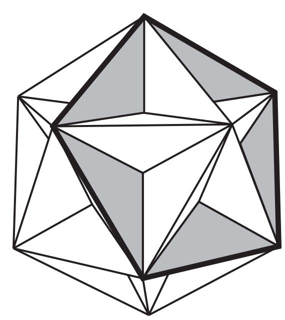
The five Platonic solids are the only five convex regular polyhedra. But if we allow nonconvex solids with faces that pass through each other, and faces that are self-intersecting polygons, then there are four additional regular polyhedra called the *Kepler-Poinsot polyhedra.*

1. Make a b_3 regular icosahedron.

Q1 Look at the Zome regular icosahedron and its 30 edges. Try not to see them as forming triangles; instead, see the edges of b_3 regular pentagons. Imagine these pentagons as surfaces forming a polyhedron. The pentagons pass through each other. How many pentagons are there altogether? How many meet at each edge? How many meet at each vertex?

The edges and vertices of the icosahedron are the same as those of a nonconvex regular polyhedron consisting of 12 pentagons that pass through each other. To distinguish it from the ordinary dodecahedron, which also consists of 12 pentagons, this polyhedron is called the *great dodecahedron.* It is called *great* because it contains a smaller dodecahedron within it.

This figure shows how the faces intersect, with only parts of each pentagon visible from the exterior. There are only 30 edges—the icosahedron edges. The other lines are lines of intersection, where faces cross and continue on the inside. To understand this object, mentally connect five visible isosceles triangles into a largely hidden pentagon; one such pentagon is shaded.



Great dodecahedron

There are three different approaches to Zome models of the great dodecahedron.

a. The icosahedron shows all the edges and vertices of the great dodecahedron, so it is also a model of the great dodecahedron.

b. A second type of model also shows the lines of intersection that would be visible where pentagonal faces intersect each other. It mimics what one sees in a paper model, or the preceding figure.

c. The third type of model shows all the lines of intersection where faces intersect, including those that would be hidden inside a paper model.

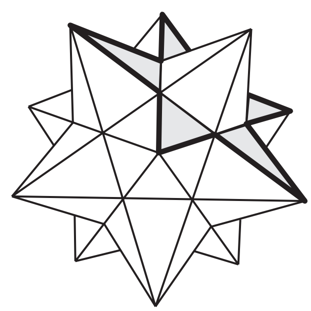
2. Complete your edge model into one of these last two types of models. Look for the small b_1 dodecahedron at the center with faces in the same planes.

Q2 Only the first type of model, which has no lines of intersection, is appropriate for counting faces, vertices, and edges of the great dodecahedron. Does Euler's formula hold for this polyhedron?

3. To make the next Kepler-Poinsot polyhedron, the *small stellated dodecahedron,* make a b_1 dodecahedron, and erect a b_2 pyramid on each face. (Or, equivalently, take the third type of great dodecahedron model above and remove its b_3 struts.)

One way of seeing these edges is as an *elevated dodecahedron.* That is an appropriate name if you think of it as a non-self-intersecting polyhedron of 60 isosceles triangles with 32 vertices, a dodecahedron that has had a pyramid erected on each face.

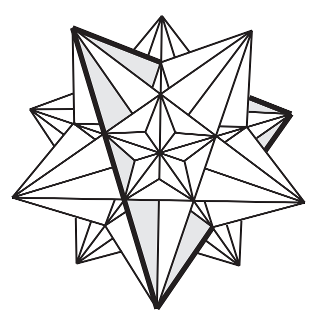
Q3 Look at the same set of edges in a new way, as the *small stellated dodecahedron,* a self-intersecting polyhedron in which every face is a pentagram. How many faces, vertices, and edges does it have? Again, Euler's formula is not expected to apply.



Small stellated dodecahedron

Q4 There is another way to see this same set of edges as yet a different self-intersecting polyhedron, the *great icosahedron.* Find it and describe its structure. (What kind of faces? How many per vertex? How many per edge? How many vertices and edges?) (Hint: Find an equilateral triangle among the edges.)

The shading in the figure illustrates one face plane of the great icosahedron. The lines of intersection between the triangular faces would be green struts. You cannot construct this model in the model of the small stellated dodecahedron because five green struts would have to go into one red hole.

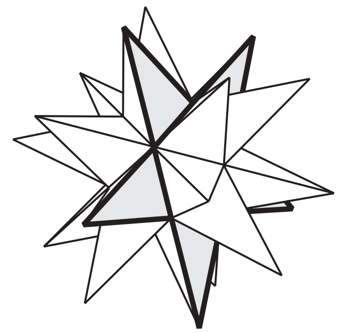


Great icosahedron

4. Make a b_1 icosahedron. Erect a b_2 pyramid on each face.

   Q5 See the structure just formed as a self-intersecting polyhedron composed of pentagrams. Describe its structure.

   Q6 If *X* and *Y* are a pair of dual polyhedra, then how are the numbers of faces and vertices of *X* related to the numbers of faces and vertices of *Y*?



Great stellated dodecahedron

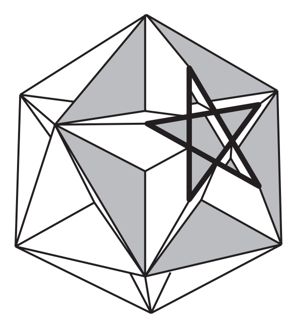
   Q7 Which of the four Kepler-Poinsot polyhedra are dual pairs?

To better understand the duality of these self-intersecting forms, use the notation $\{n, m\}$, introduced in Unit 3, to refer to a regular polyhedron consisting of *n*-gon faces, *m* at each vertex.

   Q8 What is the $\{n, m\}$ notation for the small stellated dodecahedron and the great stellated dodecahedron?

   Q9 Recall from Unit 3 that $\{n, m\}$ and $\{m, n\}$ are a dual pair. Trusting that this generalizes and considering the dual pairs discovered in Question 7, what should be the notation for the great dodecahedron and the great icosahedron?

To make sense of a notation like $\{5, 5/2\}$, you need to think about the notation $\{n, m\}$ and look at the vertices of the great dodecahedron and great icosahedron. There are five pentagons at each vertex of the great dodecahedron, but they are not arranged in a single cycle, like the five triangles of an icosahedron. Instead, they go around twice—if you truncate the corner, you will see a $5/2$-gon cross section. This is what the second position in this notation means. Similarly, the five triangles at any vertex of the great icosahedron go around twice.



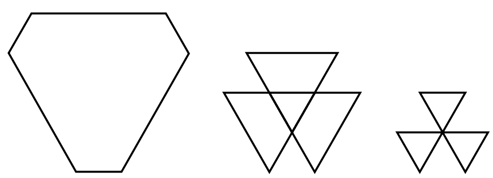
Cross section of a great dodecahedron

20.3 Uniform Polyhedra

Challenge

Imagine a turtle moves forward 30 steps, turns right 120 degrees, moves forward 10 steps, turns right 120 degrees, and repeats this sequence until it is back at its starting point. What figure does it trace? Build a Zome model.

1. Using b_1s only, build the three structures shown. Each is planar, with 60-degree and 120-degree angles. The lengths of the line segments are b_1 and $3b_1$ in the first, $2b_1$ and $3b_1$ in the second, and b_1 and $2b_1$ in the third.



Uniform 6-gons

Q1 Describe how each of these models can represent a kind of hexagon with six vertices. Explain how there are two lengths of edges in each, but only one type of vertex. Which balls represent vertices?

Look at the uniform 6-gons in the figure as polygons formed by a sequence of vertices and edges, not as a figure enclosing a particular region of the plane.

2. Pick any two vertices in a uniform 6-gon and show how the model can be rotated or reflected so that the first vertex moves to the position of the second while the model as a whole appears unchanged.

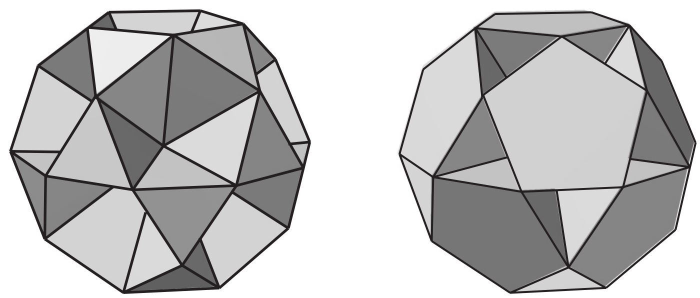
Like uniform polygons, *uniform polyhedra* have only one kind of vertex, and any vertex can be moved to the position of any other by a symmetry operation (rotation or reflection) that leaves the polyhedron as a whole appearing unchanged. Uniform polyhedra may have more than one type of face, just as uniform polygons may have more than one length of edge. The faces of a uniform polyhedron may pass through each other or through the polyhedron's center, just as the edges of the 6-gons above pass through each other or through the polygon's center.

The faces of a uniform polyhedron are regular polygons, possibly self-intersecting. Special cases of the uniform polyhedra, in which the faces do not pass through each other, are the Archimedean solids, some prisms, and some antiprisms. Special cases in which there is only one type of face are the Kepler-Poinsot polyhedra. The most special cases are the Platonic solids.

3. Make at least half of an icosidodecahedron (3, 5, 3, 5).

Q2 Look at it in a new way, which involves larger faces that intersect each other. See a polyhedron in which some faces go directly through the center. What are those faces? Using the figure as a guide, describe one self-intersecting polyhedron.

Q3 Using the figure as a guide, describe a different self-intersecting polyhedron with the same edges as the icosidodecahedron.

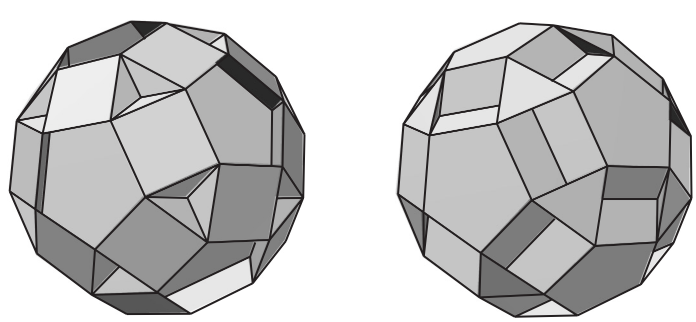


(3, 10, 3, 10) (5, 10, 5, 10)

Some self-intersecting polyhedra

4. Make at least a third of a rhombicosidodecahedron (3, 4, 5, 4).

Q4 See its edges in a new manner, which involves larger faces that intersect each other. Again, there are two different self-intersecting polyhedra with the same edges. Using the figure as a guide, can you find them both? Remember that exactly two faces must meet at each edge.

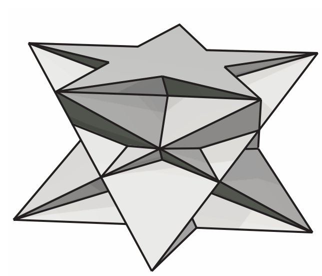


(4, 10, 4, 10) (3, 10, 5, 10)

More self-intersecting polyhedra

There are 53 uniform polyhedra, not counting the Archimedean or regular polyhedra. In addition, there are infinite families of prisms, antiprisms, and *crossed antiprisms,* in which the triangular faces cross the central *n*-fold axis.

5. Make a $b_2 + b_1 + b_2$ pentagram. Make one equilateral triangle with edge $b_2 + b_1 + b_2$. The pentagram and the equilateral triangle are the two types of faces in the polyhedron you will now construct. Place the pentagram flat on the table and hold the triangle slanted above it, with one edge parallel to one pentagram edge, and with the triangle crossing the vertical five-fold axis. If properly arranged, the zomeballs will be parallel, so the flat sides of the parallel blue edges will face the same way. If they are not parallel, turn over the pentagram or the triangle. When you find the proper arrangement, remove one edge of the triangle and add the rest of it to the pentagram, slanting up. Build four more triangles so there is one from each of the pentagram's five edges. Each triangle will intersect two other triangles. Their tops will provide the vertices for the second pentagram base.



Pentagrammatic crossed antiprism

Study your model of the *pentagrammatic crossed antiprism,* and see it as ($5/2$, 3, 3, 3), consisting of two pentagrams and ten triangles. At any vertex of the base, there are two triangles that point up—that share an edge with the lower pentagram. But they do not share an edge with each other. The gap between them is filled by a triangle that points down. (Its third edge is shared with the upper pentagram.) The polyhedron has the same symmetry as a pentagonal antiprism. Its zigzag of edges forms a self-intersecting skew 10-gon. As the figure shows, the triangular faces cut deeply through each other.

Explorations 20

A. Crowns Make a regular skew 10-gon from r_1s. (Every angle will be the same as the obtuse angle of a red rhombus.) Extend each edge in both directions with r_2s, and join them with ten new balls. The result is a type of self-intersecting skew $^{10}/_{3}$-gon, which you can wear as a kind of crown. Can you make a yellow one? How are these crowns related to crossed antiprisms?

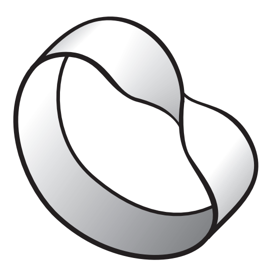
B. Dual Kepler-Poinsot Pairs Make models of concentric dual pairs for the Kepler-Poinsot polyhedra.

C. Five Cubes and Three More The compound of five cubes (see Unit 11) provides the edges (and vertices) of three more uniform polyhedra. Study a model and see the edges in new ways.

D. Uniform Polyhedra by Truncation Another technique for finding uniform polyhedra is to truncate regular polyhedra. Recall that one can truncate a vertex either partway or to the edge midpoint.

- **a.** Visualize what you get if you truncate the great dodecahedron to a depth that its pentagon faces become regular 10-gons. Remember that truncating a vertex reveals a pentagram (not pentagon) as the cross section. You can make a model where each edge of the pentagrams in the result is of size $b_2 + b_1 + b_2$. You can begin by making one vertex of a $b_3 + b_4 + b_3$ great dodecahedron and truncating it by creating a $b_2 + b_1 + b_2$ pentagram.
- **b.** What if you truncate the great dodecahedron further, to its edge midpoints? Again, you can make a model where each edge of the pentagrams in the result is of size $b_2 + b_1 + b_2$. You can begin by truncating (part of) a $2b_4$ great dodecahedron.
- **c.** If you truncate the great icosahedron to its edge midpoints, you get the ($^5/_2$, 3, $^5/_2$, 3), with 12 pentagrams and 20 triangles. Make a Zome model of part of it.

E. One-Sided Heptahedron You may be familiar with the Möbius strip, a one-sided, one-edged surface discovered by the German mathematician and astronomer August Möbius (1790–1868). Surprisingly, there exists a one-sided heptahedron, a uniform polyhedron with seven faces and only one side. To discover it, consider the operations applied to the icosidodecahedron in Question 2 of Activity 20.3, and apply them to the regular octahedron.

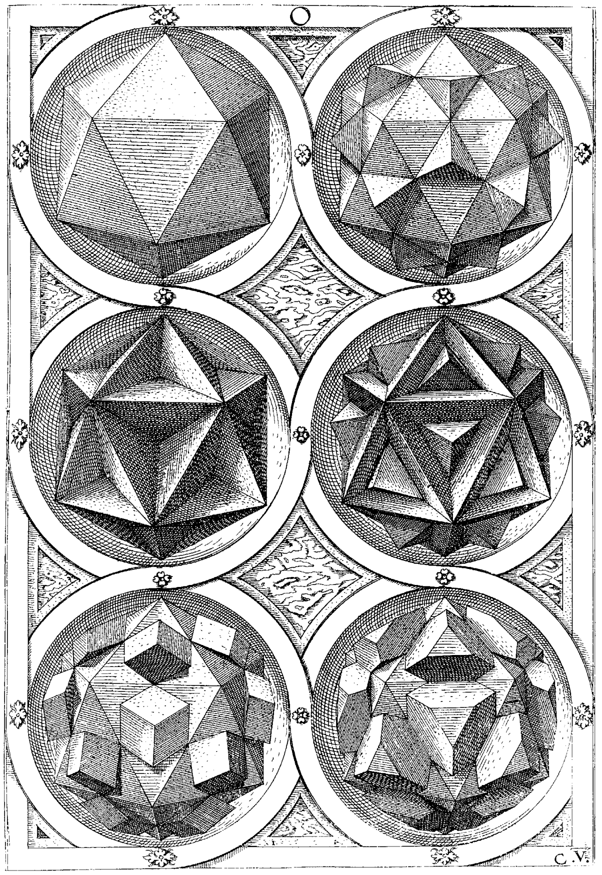


Möbius strip

F. **Pentagrammatic Concave Trapezohedron** You can make the dual of the pentagrammatic crossed antiprism. Because the uniform polyhedron has ten identical vertices, the dual has ten identical faces. Each face is a kind of arrowhead concave quadrilateral. First make one face: Make two $b_2 + b_1 + b_2$ edges. Place them into a zomeball at a 36-degree angle. Then fill the sharp empty V with a less sharp (108-degree) concave V of two b_2s. To arrange ten of these into a self-intersecting polyhedron, first make a b_1 skew 10-gon of the type that is found as the equator of a regular icosahedron. (This is the same as the zigzag of a pentagonal antiprism with equilateral sides.) Notice how the two b_1s of your arrowhead are related as two edges of the skew 10-gon. Complete each such pair of edges into an arrowhead. (You can make a compound of the dual pair in double scale. If you do, notice how respective edges are perpendicular, but some would need to be extended to actually cross.)

G. **A Self-Intersecting Rhombic Triacontahedron** It is possible to make a polyhedron bounded by 30 skinny yellow rhombi, with five meeting at each vertex. The faces and edges will pass through each other. Make skinny rhombi, each edge of length $y_1 + y_2$, with the y_1s forming the obtuse angles and the y_2s forming the acute angles. Begin the polyhedron with five of these, with their acute angles meeting at a five-fold vertex. Continue by making the other acute vertices into similar five-fold vertices. As the faces pass through each other, you will create other five-fold vertices (where five obtuse angles meet, going around twice) under the original five-fold vertices. It is a kind of nonconvex zonohedron. If you made the polyhedra in Exploration D, you can determine which nonconvex uniform polyhedron this is dual to. [Hint: In the ordinary convex rhombic triacontahedron, the vertices surrounding any face are of order 3, 5, 3, and 5, and it is dual to the icosidodecahedron, (3, 5, 3, 5).]

Connection



Some interesting polyhedra were first presented by artists. This figure shows a plate from the 1568 book *Perspectiva Corporum Regularium* by Wentzel Jamnitzer, a Nuremberg goldsmith. Notice that the middle image on the left shows the external appearance of the great dodecahedron, over 200 years before Poinsot described it and understood it mathematically as a regular polyhedron.

Courtesy of the Bancroft Library, University of California, Berkeley.

UNIT 21

The Fourth Dimension

Students are introduced to three families of *n*-dimensional generalizations of polygons and polyhedra. They build three-dimensional models of four-dimensional figures.

Goals

- To introduce the fourth dimension
- To reason by analogy

Prerequisites

Students need experiences gained from Unit 14, Zonohedra. For Activity 21.3, students need the experience gained in Unit 19, Coordinates.

Notes

Higher-dimensional geometry goes beyond three-dimensional visualization skills, so one must often work by extending patterns. Some people develop the ability to visualize higher-dimensional objects in an intuitive manner. Reassure students that they do not need to feel a direct, intuitive grasp of the objects. Mathematics is not about physical reality so much as patterns and relationships.

This unit uses analogy to explore *n* dimensions, with no attempt at completeness, merely highlighting some of the more accessible ideas in an introductory manner. In particular, *regular, convex,* and *dual* are not defined here beyond three dimensions. Students may appreciate knowing that those terms (and more) can be generalized to *n* dimensions in ways analogous to the three-dimensional case.

21.1 Hypercubes

As students build the third model of the hypercube, help them notice that the outer shell is a zonohedron. This is not a coincidence. When you follow the procedure given for making hypercubes, every face added at each stage is a parallelogram. As a consequence, the outer surface of these hypercube models is always a zonohedron.

21.2 Simplexes

The two basic approaches for dealing with *n*-dimensional geometry are analogy and coordinate methods. Just as analogy was used to generalize cubes to hypercubes, it is used in this activity to generalize tetrahedra to simplexes.

21.3 Coordinates and Cross Polytopes

In this activity, students will use coordinates to understand cross polytopes.

21.4 The 120-Cell

Once students have completed Activities 21.1–21.3, you can provide this information:

> The three families—simplexes, hypercubes, and cross polytopes—are regular convex polytopes in any number of dimensions. They are the only regular convex polytopes in five or more dimensions. In three dimensions, we know there are also two other regular convex polytopes, the icosahedron and the dodecahedron, making five altogether—the Platonic solids. Interestingly, in four dimensions, there are six regular convex polytopes. No dimension (except two dimensions, where there is an infinite number of regular polytopes, the regular n-gons) has more.
>
> We can enumerate all the convex, regular, four-dimensional polytopes in terms of the number of regular polyhedra that surround each edge. The constraint here is that the dihedral angles around an edge must sum to less than 360 degrees. You can verify that these are the only possibilities for what surrounds each edge:
>
> - 3 tetrahedra per edge—the four-dimensional simplex, or *5-cell,* composed of 5 tetrahedra.
> - 3 cubes per edge—the four-dimensional hypercube, or *8-cell,* composed of 8 cubes.
> - 4 tetrahedra per edge—the four-dimensional cross polytope, or *16-cell,* composed of 16 tetrahedra.
> - 3 octahedra per edge—the *24-cell,* composed of 24 octahedra.
> - 3 dodecahedra per edge—the *120-cell,* composed of 120 dodecahedra.
> - 5 tetrahedra per edge—the *600-cell,* composed of 600 tetrahedra.
>
> The first three possibilities are introduced in Activities 21.1, 21.2, and 21.3. The last three are complex. The 120-cell is constructed in Activity 21.4; its image is on the color insert. If you don't have enough materials to build the whole 120-cell, you can build a part of it, by building only above the base of the central dodecahedron. The 600-cell is built in Explorations 21.

For a deeper introduction to n-dimensional geometry, see *Beyond the Third Dimension,* by Thomas Banchoff. For interested students, you might recommend the book *Flatland,* by Edwin Abbott.

21.1 Hypercubes

Challenge

Make a dodecahedron in which six of the faces are squares and six are parallelograms.

Hypercubes are *n*-dimensional analogs of the cube. They are special *polytopes,* the general term for polygons, polyhedra, and so on, of any number of dimensions. A polygon is a two-dimensional polytope; a polyhedron is a three-dimensional polytope.

1. Place these four things in a row: a single zomeball, a b_1 strut with a zomeball at each end, a b_1 square, a b_1 cube.

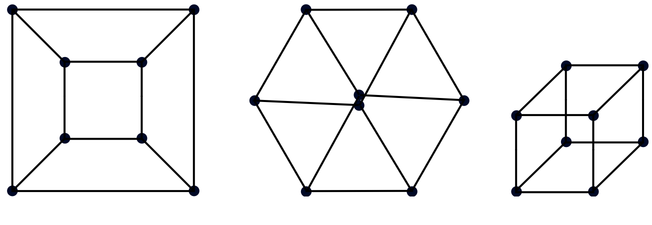
These are zero-, one-, two-, and three-dimensional hypercubes. The following procedure can be used to make an *n*-dimensional hypercube from an $(n - 1)$-dimensional hypercube: Make two parallel copies of the object that is one dimension lower, and connect all pairs of corresponding vertices with struts, using a new direction (perpendicular to all previous ones).

Q1 Find a pattern for the number of vertices in an *n*-dimensional hypercube.

Q2 Extending the pattern, how many vertices do you expect there to be in a four-dimensional hypercube?

Q3 Find a pattern for the number of edges in an *n*-dimensional hypercube. For a general formula, notice in your four hypercube models that each vertex touches *n* edges. How many edges do you expect there to be in a four-dimensional hypercube?

You cannot make a true model of the four-dimensional hypercube because you exist in a three-dimensional universe; but you can make various models of the projection of a hypercube to three dimensions. Consider how to make a two-dimensional projection of the three-dimensional cube. Three ways to do this are shown in the figure.



Three ways to draw a cube

One method puts a face center in the center of the drawing and preserves one four-fold axis of symmetry. Angles and lengths are distorted, since some squares appear as trapezoids, but no lines cross. A second method puts a vertex near the center of the drawing and preserves one three-fold axis of symmetry. Angles are distorted, since the squares appear as rhombi, but you can arrange to have edge lengths preserved. Two opposite vertices are nearly superimposed, and the figure violates the three-fold symmetry just slightly, in order to show the two vertices as distinct. A third method shows two parallel squares as identical squares, in order to keep some of the angles unchanged. You will make 3 three-dimensional models of the four-dimensional hypercube, each analogous to one of these methods.

2. Make a b_1 cube and a b_3 cube. Place the b_1 cube in the center of the b_3 cube and connect the two using eight y_2s.

   Q4 How do the numbers of vertices and edges of this first hypercube correspond to the patterns in Questions 2 and 3? How does the procedure for this model compare to the procedure for making an n-dimensional hypercube from an $(n - 1)$-dimensional hypercube?

   Q5 How is this hypercube analogous to the first drawing of a cube?

3. Make a y_3 rhombic dodecahedron. Place a zomeball at its center connected (with four y_3s) to four of the three-fold vertices in such a way that the four make as great an angle as possible with each other. (The other endpoints of these four struts are four vertices of an imagined regular tetrahedron.) Notice how each strut is a three-fold axis for the set of four and how the rhombic dodecahedron has been dissected into four obtuse rhombohedra. There are four remaining vertices that contact only three struts. Connect each of them to a second zomeball at the center. The two zomeballs near the center will be pushing against each other, but the struts are flexible enough to allow this slight distortion.

   Q6 How does the second version of a hypercube correspond to the patterns in Questions 2 and 3? How does the procedure for this model compare to the procedure for making an n-dimensional hypercube from an $(n - 1)$-dimensional hypercube?

   Q7 How is this second hypercube analogous to the second drawing of a cube?

4. Make two interlocking cubes of the same size connected by eight parallel diagonal lines. (If you have trouble connecting the two cubes, check that the corresponding struts have the same orientation.)

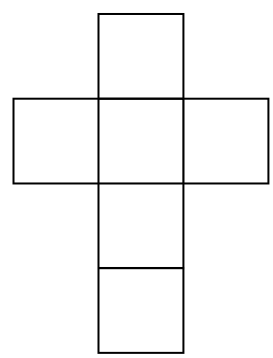
Q8 How is this third hypercube analogous to the third drawing of a cube?

Q9 How does your answer to Question 8 answer the Challenge?

Q10 Review the sequence of hypercubes and notice: Two points bound a segment. Four segments bound a square. Six squares bound a cube. What comes next in this sequence?

To build a cube, one can think of placing three squares around every vertex. The analogous process—with everything shifted up a dimension—is that to build a four-dimensional hypercube, one thinks of placing three cubes around every edge. The three-dimensional polyhedra assembled into a four-dimensional polytope are called its *cells.* In a polyhedron, two faces meet at each edge; in a four-dimensional polytope, two cells meet at each face. The cube is possible because the three squares' vertex angles sum to less than 360 degrees when laid flat in a plane; so in three dimensions, they can be rotated (about edges) to join together. The four-dimensional hypercube is possible because the three cubes' dihedral angles sum to less than 360 degrees if put together around an edge in 3-space, so in 4-space they can be rotated (about faces!) to join together. In our three-dimensional models, of course, the dihedral angles are distorted to fit in 3-space, but you can see that each edge is a part of three different cubical cells.

For a final model of the hypercube, you will break some of its connections and unfold it so that its cells lie flat in three dimensions. First consider that the *net* of the cube shows six undistorted squares in a plane. It is understood that some of the cube's edges were cut and appear as two separated square edges in this representation. Similarly, some of the cube's vertices appear as two or three points in the figure. The edges are folded and taped to put the cube back together.



Net of unfolded cube

5. Make a blue cube. Surround it by six more blue cubes, one attached to each face. Rest it on any square face, and build one more cube on the top square face.

Q11 How is this model a four-dimensional analog to the unfolded cube?

21.2 Simplexes

Challenge

Make a structure using exactly five zomeballs and ten struts, in which every pair of balls is directly connected with a strut. Then try six, and then seven zomeballs, with every pair connected.

Simplexes are *n*-dimensional generalizations of the tetrahedron. At each dimension, they are the simplest possible polytope.

1. Place these four things in a row: a single zomeball, a g_1 (or b_1) strut with a zomeball at each end, a g_1 (or b_1) equilateral triangle, and a g_1 regular tetrahedron (or a b_1 and r_1 tetrahedron).

These are zero-, one-, two-, and three-dimensional simplexes. To make each item from the previous one, add a new vertex that connects to each of the other vertices with new struts, keeping the figure as symmetric as possible.

Q1 Find a pattern for the number of vertices in an *n*-dimensional simplex.

Q2 How many vertices do you expect there to be in a four-dimensional simplex?

Q3 Find a pattern for the number of edges in an *n*-dimensional simplex.

Q4 How many edges do you expect there to be in a four-dimensional simplex?

2. Make a regular tetrahedron and use yellow struts to connect its vertices to a fifth ball at the center. (If you don't have green struts, make a tetrahedron of b_3s and r_3s and use three b_2s and a y_2 to connect its vertices to the center zomeball.) Notice that each edge is surrounded by three tetrahedra and that there is an edge connecting every pair of vertices.

Q5 How does this model fit the procedure for making each simplex from the previous one and the patterns you found in Questions 1 and 3?

Q6 Two points bound a segment. Three segments bound a triangle. Four triangles bound a tetrahedron. What comes next in this series?

Q7 A two-dimensional drawing of the four-dimensional simplex will consist of five dots, with a line connecting every pair. What is the most symmetric way to draw this?

Q8 In that drawing, count the tetrahedra. Verify that three distinct ones share each edge and that four distinct ones meet at each vertex.

3. Build an unfolded model of the four-dimensional simplex. The Zome System does not have the angles to build this exactly, but you can make an approximation.

21.3 Coordinates and Cross Polytopes

Challenge

What is the dual to the four-dimensional hypercube? (Hint: To draw the dual of a four-dimensional polytope, place a vertex in the center of each cell, and draw an edge connecting vertices whenever two cells are adjacent.)

A square of edge length 2, centered on the origin, has its four vertices at (1, 1), (1, –1), (–1, 1), (–1, –1). These four coordinates can be written as (±1, ±1). The coordinates of the eight vertices of a cube of edge length 2, centered on the origin, are (±1, ±1, ±1). The pattern continues. (For a four-dimensional point, use four coordinates.)

Q1 Give several pairs of vertices of such a cube that are connected by an edge. Give several that are not. How can you tell at a glance whether a given pair belongs on the first or second list?

Q2 Given two vertices of a hypercube, how can you tell if they are connected by an edge? (Hint: Every edge is parallel to one of the axes.)

Given two points in four-dimensional space, (w_1, x_1, y_1, z_1) and (w_2, x_2, y_2, z_2), you can define the distance between them, by analogy from the two-dimensional and three-dimensional formulas, as

$$\sqrt{(w_1 - w_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Q3 Pick any two edges of the four-dimensional hypercube with vertices (±1, ±1, ±1, ±1) and find their length.

Q4 How long is the longest diagonal of a four-dimensional hypercube?

Cross polytopes can be described by the coordinates of their vertices. A cross polytope has all of its vertices on the coordinate axes, at unit distance from the origin. So the two-dimensional cross polytope is a square, with vertices (±1, 0) and (0, ±1). The three-dimensional cross polytope has vertices at (±1, 0, 0), (0, ±1, 0), and (0, 0, ±1). (There is one vertex in the center of each face of the cube of side length 2 centered on the origin.) Every vertex connects to all the others except the one opposite it on the same axis.

Q5 Which polyhedron is the cross polytope of three dimensions?

Q6 How many vertices and edges are there in an n-dimensional cross polytope?

Q7 How long is an edge of the n-dimensional cross polytope? Assume n-dimensional coordinates generalized from the cross polytopes above.

With more analysis it can be shown that the four-dimensional cross polytope consists of 16 regular tetrahedra, joined so that four surround each edge. It is the dual to the hypercube. See Exploration C.

21.4 The 120-Cell

Challenge

Find as many Zome pentagons as you can that have mirror symmetry.

The 120-cell is a regular convex four-dimensional polytope composed of 120 regular dodecahedra joined three around each edge. The following projection of this solid into three dimensions is centered on one undistorted dodecahedron, surrounded by other dodecahedra that are flattened. There are five layers of dodecahedra, each more flattened than the layers inside it.

To prepare for building the 120-cell, make one of each type of dodecahedron.

1. Make a b_2 regular dodecahedron. This is the central cell of the model.
2. Make a slightly flattened dodecahedron consisting of two opposite b_2 pentagons, an r_2 zigzag equator, and ten y_2s connecting the equator to the pentagons. This dodecahedron has one axis of five-fold symmetry. Ten of its faces are of the form b_2–y_2–r_2–r_2–y_2. (If you have trouble, try turning over your first pentagon.)
3. Make a slightly more flattened dodecahedron with a three-fold axis of symmetry. Join three b_2–y_2–r_2–r_2–y_2 pentagons together around one vertex, by sharing their red edges. In the three places where two y_2s touch, make a y_2–y_2–r_1–b_2–r_1 pentagon. (Keep each pentagon in the plane defined by its two y_2s.) This gives half the dodecahedron; make the other half so that it is symmetric, with a three-fold axis.
4. Make a rather flattened dodecahedron with five-fold symmetry, consisting of two opposite b_2 pentagons, a y_2 zigzag equator, and ten r_1s connecting the equator to the pentagons. Ten of its faces are of the form y_2–y_2–r_1–b_2–r_1.
5. Make a planar construction consisting of two b_2–y_2–r_2–r_2–y_2 pentagons sharing the b_2. In the two y_2–y_2 concavities, add two r_1s and a b_2 to make y_2–y_2–r_1–b_2–r_1 pentagons. The result is a flat construction of four pentagons. Four of its exterior sides are of the form $r_1 + r_2$.

These five models are progressively more flattened dodecahedra. The last is totally flattened and shows only four faces. Think of it as a dodecahedron standing on a edge (two-fold axis vertical) squashed so that the top four faces lie above the bottom four and the four side faces are collapsed into one line each (the $r_1 + r_2$ lines).

6. Hold the regular dodecahedron so that you see it like the fifth, completely flat construction.

Once you understand these five components, you can build the 120-cell with components meeting pentagon-to-pentagon. The whole has icosahedral symmetry; the symmetry axes of the components are lined up with the axes of the whole.

Do not build a number of the components and attempt to join them. Instead, keep them in mind as you build piece by piece. Start at the core and work outward, layer by layer, turning the partial structure over as you work on each layer.

7. Use the regular dodecahedron, type 1, as the central core. Surround it with 12 dodecahedra of type 2. In each of the concavities on three-fold axes, build a dodecahedron of type 3. This creates, on the five-fold axes, new cavities that are filled in by dodecahedra of type 4. Finally, you will see that the outer surface is covered with flat hexagons of type 5.

Recall that all the cells are regular dodecahedra in the four-dimensional object. They appear distorted because you flattened the model to fit in three dimensions.

To account for the 120 dodecahedra, you must interpret the model as folding over itself. Each of the dodecahedra of types 1 to 4 represents two overlapping dodecahedra that happen to coincide. This is analogous to projecting a cube along one of its edge directions so that two opposite faces exactly overlap and one sees only a square. Four sides of the cube would be flattened into lines, which sit on the outer four sides of the square. Here, the flat dodecahedra of type 5 correspond to the squares flattened into lines. They are not doubled; they are what join the two overlaying structures. With this understanding of the exterior surface, one can also verify that three dodecahedra surround each edge and four surround each vertex.

Q1 Fill out the table, and check that 120 cells are represented:

Cell type	Number in model	Number of four-dimensional cells represented
1		
2		
3		
4		
5		
Total	—	120

Explorations 21

A. 5D Hypercube Make a model of a hypercube in five or more dimensions.

B. 5D Simplex Make a model of a five-dimensional simplex. Then try a six-dimensional simplex. (Hint: Use 7 balls and 21 struts connected so that every pair is directly connected with no struts crossing; some green struts are needed.)

C. Tetrahedra in the Four-Dimensional Cross Polytope There are 16 regular tetrahedra in the four-dimensional cross polytope. Where are they? To find one tetrahedron, find four equidistant vertices.

D. The 600-Cell A model of the 600-cell can show how five tetrahedra surround each edge. At its core is a b_2 icosahedron divided into 20 almost-regular tetrahedra. To its exterior, add another layer of tetrahedra using y_2 slanting edges. That makes two layers of tetrahedra centered over the three-fold axes. Connect the new vertices to each other with b_2s, making a third layer: 30 tetrahedra over the two-fold axes. With r_1s and r_2s, add 60 more tetrahedra in groups of five around the five-fold axes. This produces a rhombic triacontahedron. On the triacontahedron's rhombi, add r_1s and y_2s as the slanting edges of 30 rhombic pyramids, that is, 60 tetrahedra. Then connect their peaks using 60 b_2s (that outlines an icosidodecahedron) to complete 60 tetrahedra around the 5-fold axes and 20 over the 3-fold axes. The missing tetrahedra are all flattened on its surface.

E. Another Hypercube Projection You made three-dimensional models of the four-dimensional hypercube that were centered on a cube and centered on a vertex. Find a symmetric model centered on an edge. (Hint: Two opposite edges will end up pushing against each other at the center of the model.)

F. Euler's Theorem in Four Dimensions Based on the simplex and hypercube, conjecture an analog to Euler's theorem for four-dimensional polyhedra.

UNIT 22

Stellations

This is a hands-on introduction to stellated polyhedra.

Goals

- To introduce stellations
- To look in a new way at polyhedra students have built

Prerequisites

Students will have encountered some of these models in Units 2, 3, 5, 12, and 21. They need to be familiar with the rhombic dodecahedron and the rhombic triacontahedron. The Challenge for Activity 22.1 assumes familiarity with Unit 19.

Notes

The word *stellate* is from the Latin for *star.* To make a polyhedron that is starlike, start with any polyhedron and erect *any* pyramid on its faces. A few authors call this a *stellation,* but it is more accurately called an *elevated polyhedron.* For a stellation, the face planes of the core polyhedron are extended. (A few other authors prefer a stricter sense and define a stellation as a polyhedron that extends the edges of the core polyhedron.)

22.1 Stellated Polyhedra

Green struts are required. The polyhedron in Exercise 1 was called a *concave equilateral deltahedron* in an exploration in Unit 2, but here you see it in a new light. The word *enantiomorph,* used in Exercise 3, was defined in Unit 5 as one of the two mirror images of a chiral object. For pictures and discussion of all the stellations of the icosahedron, see *The Fifty-Nine Icosahedra,* by H. S. M. Coxeter.

22.2 More Stellations

In the activity, students construct all three stellations of the rhombic dodecahedron and two of the many stellations of the rhombic triacontahedron. The compound of five cubes, introduced in Unit 11, is seen in a new light as a stellation of the rhombic triacontahedron, as is a self-intersecting zonohedron from Exploration 20G.

22.1 Stellated Polyhedra

Challenge

The small stellated dodecahedron $\{5/2, 5\}$, the great dodecahedron $\{5, 5/2\}$, and the great stellated dodecahedron $\{5/2, 3\}$ built in Unit 19 each has 12 faces. If each face was made of an opaque material and you sat at the center (with a flashlight), what would you see?

Given a polyhedron *X,* a "stellated *X*" has five properties:

- It has an *X* at its core.
- It has the same number of faces as *X*.
- Its faces are in the same planes as the faces of *X*.
- It has the same symmetry as *X*.
- All of its faces are identical.

Some stellated polyhedra can be created by building a pyramid on each face of a given polyhedron, taking care that the height of the pyramids is chosen such that their faces lie in the planes of the original faces.

Q1 The *stella octangula* can be seen as a stellation of what polyhedron? Verify the five properties.

In a stellation, one *face* may consist of several disconnected facets in the same plane. Here is an example where each face is disconnected and where the stellation does not involve pyramids on the faces of the polyhedron that is being stellated:

1. Build at least half of a regular dodecahedron. Then erect pyramids on the inside of each face. Use only one strut length so that the result can be seen as 60 equilateral triangles.
2. Find three equilateral triangles that are in one plane. The group of three triangles is considered one face.

Q2 Thinking of three disconnected triangles in a plane as a single face, how many faces are there? Each of these faces is on the plane of a face of a certain Platonic solid. Visualize this solid, and name this stellation.

In many cases there are different stellations possible from one initial polyhedron; for example, there are 59 different stellations of the icosahedron. In Exercise 3, once again, each of the 20 faces is an assemblage of disconnected triangles.

3. Build a b_3–b_2–g_2 triangle and a b_2–g_1–g_2 triangle, both in the yellow plane. Construct an irregular tetrahedron consisting of two of the first type and two of the second type. (It can be made in either of two enantiomorphs.)
4. Build a b_3 dodecahedron. Using each dodecahedron edge, build one of these irregular tetrahedra inside the dodecahedron—all 30 of the same handedness. They contact each other only at the dodecahedron vertices.
5. Study this form and see how every facet is in the plane of an imaginary central icosahedron.

   Q3 Make a sketch of one face. (Hint: There are six triangles in a face.)

   Q4 What polyhedron do the vertices on the interior of the model outline?

Another stellation of the icosahedron is created by building a low pyramid with green slanting edges over each face of the icosahedron. For the lengths to work out, you start with an icosahedron of edge $b_1 + b_3$.

6. Make a pyramid with an equilateral $b_1 + b_3$ triangular base and g_2 slanting edges. Build three more of these around the first so that the blue triangles meet as faces of an icosahedron.

This is a portion of a stellated icosahedron. You can see that the plane of the central triangle is extended into an irregular green 6-gon.

7. Make one of these planar, green, irregular 6-gon faces. You can use any size of green strut.

This is one face of *the first stellation of the icosahedron,* which consists of 20 of these 6-gons, passing through each other. Its edges are arranged like those of the rhombic triacontahedron, but the green "rhombi" are not planar.

8. Make a complete model, building the green portion of this structure, omitting the blue. Use any size of green strut.

22.2 More Stellations

Challenge

Invent a stellation for a polyhedron of your choice.

Given a polyhedron, imagine one of its face planes extended. Imagine the lines that are created when that one face plane is intersected by all the other face planes of the polyhedron. The faces of the stellation must be a shape outlined by some of those lines. You will now build three stellations of the rhombic dodecahedron.

1. Make a rhombic dodecahedron. (It consists of 12 fat yellow rhombi.)
2. Imagine a rhombic pyramid formed by using two blues and two yellows to connect the four vertices of one face to the polyhedron's center. Create the first stellation of the rhombic dodecahedron by erecting this rhombic pyramid on the outside of each of the 12 faces. Each of the exterior faces of this nonconvex form is a continuation of one of the original rhombic dodecahedron's planes.
3. You can add more of these rhombic pyramids to make a second stellation. (Hint: Use the blue struts from the first stellation so that only yellow struts are needed, two for a pyramid base and one for the remaining edge.)

   Q1 How many additional rhombic pyramids are needed at this stage?
4. You can add yet more of these rhombic pyramids to make a third stellation. (Hints: Use one existing blue strut and one new blue strut for each pyramid. The new ones are *not* along the four-fold axes.)

   Q2 How many additional rhombic pyramids are needed at this stage?

There are millions of stellations of the rhombic triacontahedron. You will build two of them—one yellow and one blue.

5. Start with an $\mathbf{r_1}$ rhombus. Build a thin $\mathbf{y_3}$ rhombus in the same plane, such that each of its obtuse vertices is an acute vertex of the red rhombus.
6. Stellate an $\mathbf{r_1}$ rhombic triacontahedron by building such yellow rhombi around each of its faces. The yellow rhombi will intersect each other, so you need to build their sides as $\mathbf{y_1} + \mathbf{y_2}$ so that the $\mathbf{y_1}$s share their vertex with the $\mathbf{r_1}$s.

   Q3 Verify that all the conditions for stellations are met by this model and that the final construction is a polyhedron, with two faces meeting at each edge.

7. Make an r_2 rhombus and a $b_2 + b_1 + b_2$ square. Place the rhombus inside the square and connect its acute vertices to the square's vertices with y_3s. If you have trouble, rotate the square 90 degrees so that the zomeballs are parallel.
8. Stellate an r_2 rhombic triacontahedron by building such squares around each of its faces.

   Q4 Notice the pentagrams and count the cubes in the finished structure.

Explorations 22

A. Cube and Tetrahedron Stellations Explain why there are no stellations of the cube or tetrahedron.

B. Stellating the Dodecahedron The great dodecahedron, the small stellated dodecahedron, and the great stellated dodecahedron (see Unit 20) are all stellations of the dodecahedron. Explain why. Build all three simultaneously around one core dodecahedron.

C. Stellated Cuboctahedron and Icosidodecahedron Construct a cuboctahedron and erect low square or triangular pyramids on each of its faces. Choose the height of each pyramid such that each new triangular face continues the plane of the adjacent cuboctahedron face. What is the result? Repeat, starting with an icosidodecahedron.

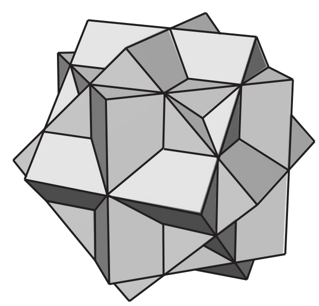
D. A Red Rhombic Triacontahedron Stellation Stellate an r_1 rhombic triacontahedron by building a concentric and parallel $r_2 + r_1 + r_2$ rhombus around each of its faces. Use b_2s as scaffolding to connect the obtuse vertices of each r_1 rhombus to the parallel $r_2 + r_1 + r_2$ rhombus. Verify that all the conditions for stellations are met by this model and that the final construction is a polyhedron.

E. Another Stellation The pointy ends of 20 acute red rhombohedra can be assembled around a point to form a starlike polyhedron. Just start with a red starburst and complete each group of three struts into a rhombohedron. The rhombohedra meet face-to-face, leaving only half the rhombi visible for the exterior faces. Keeping only the exterior structure gives a structure of 60 rhombi that is a stellation of what polyhedron? (Notice it has zones, like a zonohedron.)

F. Five Tetrahedra The compound of five tetrahedra (see Unit 11) is another one of the stellations of the icosahedron. Rebuild that model and see it in this new light.

G. Five Octahedra Dual to the compound of five cubes in a dodecahedron is the compound of five octahedra around an icosahedron. Recall that in Unit 11 one octahedron was constructed around an icosahedron and that the octahedron's faces were seen to lie in the face planes of the icosahedron. The five octahedra together form a stellation of the icosahedron. The three edge lengths required can be shown to be in the ratio 2τ, τ^2, and $1 + \tau^2$. With strut lengths g_1 and g_2, these are formed as $2g_2$, $g_1 + g_2$, and $2g_1 + g_2$, so a large supply of greens is necessary to complete this model. Start by making the first stellation of the icosahedron, but in edge length $2g_2$. Above each of the nonplanar "rhombic" openings, erect a pyramid that is one corner of an octahedron. Use the $g_1 + g_2$ lengths above the short diagonal and the $2g_1 + g_2$ lengths above the long diagonal. They meet at 60- and 90-degree angles, as in the octahedron. How do five octahedra, with eight faces each, combine to give only 20 face planes?

H. Compound of Three Cubes In M. C. Escher's picture *Waterfall,* one tower is topped with the first stellation of the rhombic dodecahedron and the other is topped with a compound of three cubes shown below. To understand the compound of three cubes, imagine three identical cubes superimposed and imagine the 3 four-fold axes that they all share. Imagine gluing each cube to one of the four-fold axes. Now turn each cube 45 degrees about its axis. The result has all the symmetry of a cube. To make a Zome model, recall that in the blue plane, a green square is rotated 45 degrees from a blue square. So each cube has two green faces, connected with four blue edges. An edge length of $g_1 + g_2 + g_1$ allows for the crossings at approximately the regular 8-gon points. However, the blue edges then require struts of length $\frac{g_1 + g_2 + g_1}{2}$ and there is no blue edge of that length. But b_3 is a very close approximation, and this makes a beautiful model, in which the distortion is imperceptible. How close is the approximation to the cube?



Compound of three cubes

UNIT 23

Fractals

Students start with flat fractals, are introduced to the idea of fractal dimension, then move on to fractal models that exist in three-dimensional space.

Goals

- To look at finite approximations of fractals
- To develop a deeper understanding of dimensions

Prerequisites

To complete the whole unit, students need to understand scaling area and volume (Units 15 and 17) and logarithms.

Notes

The models in Activities 23.2 and 23.3 are fascinating and well worth building. They should generate much discussion about infinity.

If your students do not understand logarithms, you need not skip this whole unit. You might skip Activity 23.1 and the questions on fractal dimension in the subsequent activities. Or you might have students find the fractal dimension with a graphing calculator or graphing software, without referring to logs. For example, to answer Question 3 in Activity 23.1, they can graph $y = 3^x$ and $y = 4$ and look for their intersection. The value of x at the intersection is the solution to the equation. Or they could get a good approximation to the solution by iterative guess-and-check.

Even though some examples of fractal figures were studied in the nineteenth century, fractal geometry took off with the advent of the computer. You might complement this unit with work on the computer, using specialized software or a computer language that supports graphics.

23.1 Introduction to Fractals

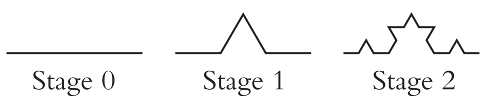
Challenge

Build a model of a pentagram inscribed in a larger pentagram. Can the pattern be continued?

Fractals are objects with *scaling symmetry.* Symmetrical polygons and polyhedra appear unchanged when rotated or reflected. Fractals have the property of appearing unchanged when looked at with a magnifying glass. A small portion of a fractal, when enlarged, looks like the whole thing.

No physical object is a true fractal, because if one magnifies it sufficiently, one sees atoms, which have their own structure. So a model of a fractal can only have scaling symmetry over a finite range of magnifications. One must imagine the finer detail, just as with other types of geometric models one must imagine, for example, that points are infinitely small, lines are infinitely long, and planes are exactly flat.

1. Make Stages 0, 1, and 2 of the Koch curve using b_1s. All angles are 60 or 120 degrees. Notice that Stage 2 is made of four copies of Stage 1. (If you make four separate Zome parts and put them together, you may have to turn some over.)



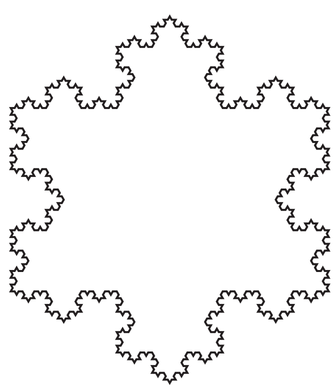
Stages 0, 1, and 2 of the Koch curve

Stage 1 is also called a *generator.* Stage 2 can be thought of as resulting from Stage 1 using a transformation rule: Each of the four edges in Stage 1 is replaced with a copy of the generator. For Stage 3, take each edge in Stage 2 and replace it with the generator. This is most easily accomplished by making four copies of Stage 2 and assembling them together.

2. Make the fourth stage of the Koch curve.

   Q1 How many struts are used in the *k*th stage?

Three copies of the fourth stage put together make a nice snowflake shape with six-fold symmetry.



Snowflake curve

In these Zome models, the overall size increases at each stage. If you had shorter struts, you would use them instead at each stage to keep the overall scale factor for length unchanged.

Q2 For the end-to-end length to remain unchanged in going from Stage k to Stage $k + 1$, the struts used at Stage $k + 1$ would be what fraction of the length of the struts at Stage k?

If you could continue this indefinitely, the result would be a fractal with infinitely many little wiggles.

The wiggles make the curve appear thicker than ordinary curves such as lines or arcs. But it is not so thick as to fill an area. There is a sense in which the Koch curve is more than one-dimensional yet less than two-dimensional. You can quantify this with the idea of a *fractal dimension.*

Recall how area scales as the second power of length and how volume scales as the third power of length. If you have a two-dimensional object and scale its length by 3, the area scales by 9. If you have a three-dimensional object and scale its length by 3, the volume scales by 27. You can write the formula $q^r = s$, where q is the scaling factor for length, r is the dimension, and s is the scaling factor for the content of the object.

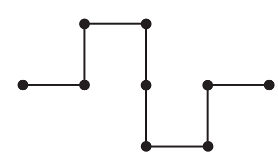
In the Koch curve, the number of struts is scaled by 4 every time the length is scaled by 3. For area and volume, the dimension—2 or 3—is the power that 3 is raised to, to get 9 or 27. For the number of struts, the fractal dimension is the power 3 is raised to, to get 4.

Q3 Three to what power is four? Solve $3^x = 4$ for x.

This curve has a fractal dimension that is between that of a line (1) and an area (2). A bumpier generator should result in a fractal with a higher fractal dimension.

3. Starting with a line segment as Stage 0 and this 8-strut generator as Stage 1, make its Stage 2.

Q4 What is the fractal dimension of the result?



Stage 1

23.2 Sierpiński in Three Dimensions

Challenge

Draw or build a large capital Y. Then replace each of its upper two branches with smaller Ys. Continue this process several levels deep.

Our next Zome fractal will be a yellow tree with three-way branches. Many natural objects, not just trees, have a structure similar to that of our fractal tree: river systems, circulatory systems, and so on.

1. Hold a zomeball and pick a yellow (triangular) hole as an axis, but don't put anything in it. Put three y_1s in the three closest yellow holes. Put a y_2 in the opposite yellow hole. If you hold it by the y_2 with the y_1s pointing upward, this is like a tree with three branches. Call it a first-level tree.
2. Make two more of these first-level trees. Assemble all three as branches into one zomeball (in three yellow holes nearest one chosen yellow hole) and add a y_3 trunk (in the hole opposite the chosen hole). If you hold it by the y_3 with the y_1s pointing upward, this is a second-level tree, with three main branches that each have three smaller branches.
3. Make two more of these second-level trees. Assemble all three as branches into one zomeball and add a y_3 trunk to make a third-level tree. (The trunk ought to be y_4, but if that is too unstable to stand up, a y_3 will do.)

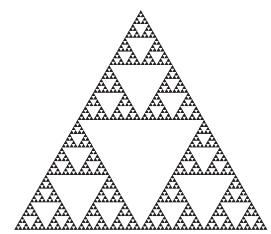
The third-level tree has 27 y_1s as its leaves. If you are careful, you can make a stand for it (put three r_1s into a ball to make a tripod) and have it stand vertically.

4. Try one of these variations:
   a. Make it a four-way branching structure by also including a y_1 in the chosen center hole in the first-level trees. Then also include a first-level tree in the center hole of the second-level trees, and so on.
   b. Choose a different branching angle in a yellow tree. Instead of the three holes closest to the chosen axis, use three (or six) near its equator.
   c. Make a five-way red tree with five r_1s and an r_2. In the third-level structure, some of the r_1 struts will get in each other's way, but just let them cross past each other.

d. Make a five-way blue tree. The trunks, being blue and not red, will not be along five-fold axes. Start with five b_1s in the five blue holes adjacent to a chosen red hole, and put the b_2 opposite one of the b_1s. The second-level tree has the form of a pentagonal pyramid. At the third level, some of the first-level balls want to be at the same position, so remove the duplicates so that the tree connects together with itself.

5. Try extending one of the above trees to the fourth level. First plan, by determining how many pieces are needed and seeing if you have enough. For structural reasons, the tree may have to rest on its branches and leaves rather than its trunk.

In 1915 the Polish mathematician Waclaw Sierpiński (1882–1969) described a triangular fractal structure that has a tetrahedral analog. Sierpiński's gasket can be obtained by removing the middle fourth of a triangle that has been divided into four smaller similar triangles, then repeating this operation on the three remaining portions, and so on.



Sierpiński's gasket

6. Make a double-scale tetrahedron. (You can use a green regular tetrahedron or the red/blue approximation. However, if you choose a tetrahedron involving several types of struts, your Zome kit will go further.)

7. Connect the midpoints of the edges to divide the tetrahedron into an octahedron and four smaller tetrahedra.

You have to imagine that the tetrahedra are solid but the octahedron is hollow.

8. Assemble four such units into a quadruple-scale tetrahedron.

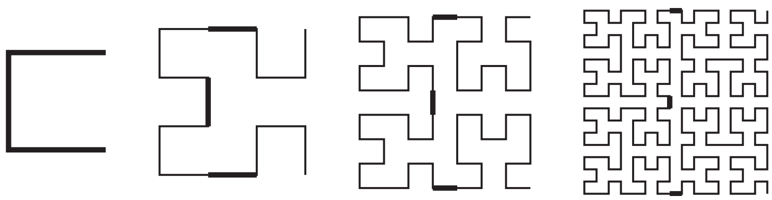
If you have time and enough pieces, you can assemble four of these into a larger tetrahedron. If you imagine keeping the overall size constant, and removing the central octahedron from ever smaller tetrahedra, you can see that at the limit, we obtain a structure that is mostly holes, so it shouldn't be surprising that the tetrahedron has dimension less than 3.

Q1 What is the fractal dimension of the structure?

23.3 More Three-Dimensional Fractals

Challenge

This fractal, at the limit, fills an entire square. Describe how to go from one level to the next.



Hilbert curve

1. Remove four edges of a $\mathbf{b_1}$ cube so that what remains is a cycle of struts that visits all eight vertices, called a *Hamiltonian cycle* after the Irish mathematician Sir William Rowan Hamilton (1805–1865). Then remove one more edge for a *Hamiltonian path* in which every vertex is visited once. There is a right angle at every turn. This is a $2 \times 2 \times 2$ structure of connected vertices.

The following steps will explain how to expand this into a $4 \times 4 \times 4$ structure in which every vertex is visited once and there is a right angle at each turn. In order to explain how each $2 \times 2 \times 2$ component is rotated, describe the three coordinate directions as L/R (left/right), U/D (up/down), and N/F (near/far). (To keep the structure stable, you may insert some temporary support struts to be removed at the end. Mark them, perhaps with a piece of tape, so that you remember which they are, or use $\mathbf{g_1}$s diagonally.)

2. Rotate your $2 \times 2 \times 2$ structure so that the path can be understood as starting at the near left lower vertex and moving in the directions U-F-D-R-U-N-D to end at the near right lower vertex.

Keep this structure in front of you as a guide: It will be the overall path in the larger structure you will build in Exercises 3–5.

3. In a separate structure, start a path that goes R-F-L-U-R-N-L — U — R-U-L-F-R-D-L. The middle U is a strut connecting two rotated copies of the $2 \times 2 \times 2$ component, one above the other, into a $2 \times 2 \times 4$ component.

4. At the end of the path, place a strut that goes in the Far direction. Then continue the path through the next two blocks of vertices as R-U-L-F-R-D-L — D — N-D-F-R-N-U-F. This gives a path in a $2 \times 4 \times 4$ block of vertices. (It will sag until it gets down to the ground again, for support.)

5. At the end of the path, place a strut that goes in the R direction. Imagine a mirror plane perpendicular to this new strut and bisecting it. Double it into a cube by building the mirror image in this plane, making a $4 \times 4 \times 4$ path.

Notice there are eight $2 \times 2 \times 2$ components all together and seven struts that connect them. The seven struts are analogous to the three darkened struts in the images of the Hilbert curve. They copy the directions U-F-D-R-U-N-D of the previous stage of the fractal, as oriented in Exercise 2. Remove any struts used for support.

Q1 How might this idea be extended indefinitely to fill a region of any size? (If you were to extend this, there would come a point [soon] when a joint would not support all the weight above it.)

If we were able to keep the overall size of the cube constant and keep shrinking the struts, how much of the space inside the cube would be taken up by the path? One way to think about this is to calculate the fractal dimension of the figure.

Q2 How many edges are in a section of the structure with $p \times q \times r$ vertices? (Include half of the strut where the blue path enters this rectangular block of space and half of the strut where the path exits.)

Q3 When you are doubling in linear dimension (from $2 \times 2 \times 2$ to $4 \times 4 \times 4$), the number of struts required is multiplied by what?

Q4 What is the fractal dimension of the structure?

The Zome components are geometric forms. They can be modeled, on a larger scale, with Zome structures! You will build the ingredients for such "mega-Zome" fractal structures.

6. As a model of a zomeball, build a polyhedron like the rhombicosidodecahedron (3, 4, 5, 4) but with b_1 pentagons and b_2 triangles. Instead of squares, it will have golden rectangles. We call that a *megaball.*

7. As a model of a blue strut, make a $b_1 \times b_2 \times 2b_3$ prism. Think of that as a *mega-*b_1 *strut*.

8. Connect the megastrut to the megaball.

9. Make a red megastrut by stacking a b_1 pentagonal prism with r_3 vertical edges, a b_1 pentagonal antiprism with b_1 zigzag, and another b_1 pentagonal prism with r_3 vertical edges.

10. Design and build a yellow megastrut.

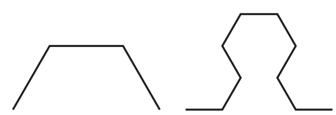
With enough pieces, you might make a mega-icosahedron.

Explorations 23

A. Fractal Pentagram A pentagram consists of five branches (72-degree–72-degree–36-degree triangles) arranged around a central regular pentagon. This would be a Stage 1 structure. Build five pentagrams, arranged around a central regular pentagon. Two sides of any pentagram correspond to two sides of a "golden triangle" arm in the next larger pentagram. This would be a Stage 2 structure. This process can be continued indefinitely, with each branch in each subpentagram being replaced by a pentagram.

B. Snowflake Perimeter and Area What is the total perimeter and area of a Koch curve-bounded snowflake at stage *n*, as *n* approaches infinity? (Assume the initial triangle has side 1.)

C. A Koch Variation Make a Koch-like curve using half a regular hexagon as generator and Stage 1. When you go to Stage 2, insert the generator first on the inside, then on the outside, then on the inside (as opposed to the Koch curve, where the generator was on the same side in all four substitutions). What is the result, what is its dimension, and how does it relate to the Sierpiński gasket?



Koch variation

D. Fractal Stella Octangula Start with a quadruple-scale tetrahedron (regular green, or red-blue approximation). Connect the midpoints of its edges to divide each face into fourths. Build four triangular pyramids with the central triangles of each face as bases. You should have a stella octangula. Now repeat the process on each triangular face. Continuing this process indefinitely creates an extremely bumpy surface.

E. Recursive Polyhedra Consider any regular polyhedron, but in the place of each of its faces (or edges or vertices) position a smaller copy of the same polyhedron. Make the smaller copies parallel and just large enough to touch their neighbors. For example, replacing each face of a cube with a smaller cube gives a cluster of six cubes around a hollow cubical center, touching along shared edges. Replacing each edge of a cube with a smaller cube gives a cluster of 12 cubes, arranged like alternate cells of a $3 \times 3 \times 3$ "Rubik's cube," touching at shared vertices. Given enough material, copies of these clusters can be used as units in larger clusters. The simplest example is the Sierpiński tetrahedron in Activity 23.2 that is just a tetrahedron in which each vertex is replaced with a smaller tetrahedron. What do you get if each edge of a dodecahedron is replaced with a smaller dodecahedron? What if each edge of an icosahedron is replaced with a smaller icosahedron?

UNIT 24

Proofs of Euler's and Descartes' Theorems

This unit develops the proofs of Euler's theorem and Descartes' theorem. For the first theorem, the topological concept of a tree is explored. For the second, special properties of triangulated polyhedra are used.

Goals

- To prove Euler's theorem
- To prove Descartes' theorem

Prerequisites

Students must remember and understand Euler's theorem (Unit 6) and Descartes' theorem (Unit 10) and have experience with duality (Unit 9).

Notes

When these theorems were introduced earlier in this book, students were guided to discover them by looking at specific examples and generalizing. Of course, that did not constitute a proof.

In this unit, we use the Zome System to help visualize several of the key concepts from topology that go into the proof. Still, at this level, we are not able to give a completely rigorous proof, and we omit certain aspects of the full proof. Specifically, the proof depends on a theorem developed by the French mathematician Camille Jordan (1838–1922). The Jordan curve theorem states that every simple closed curve on a plane surface or a sphere divides the surface into two disconnected regions (an inside and an outside if the surface is a plane). Notice that this does not hold, for example, for loops that go around the hole of a donut. Another part of the argument that is overlooked in the text is made explicit in the answers to Questions 4 and 5 in Activity 24.1.

Questions 1 and 2 in Activity 24.2 review material that was seen in Activities 4.1 and 6.2.

24.1 Proof of Euler's Theorem

Challenge

Hold the 12 vertices of an icosahedron in place, using as few struts as possible.

A *tree* is any single (connected) Zome structure with n struts and $n + 1$ balls. By *single structure,* we mean that everything connects somehow, so there are no loose pieces. Colors, lengths, and angles do not matter. Only the number of components is important in the definition.

1. Make some trees.

There are many shapes for trees. Some are like snakes with a ball at the head and tail. Some are like a starburst, having a central node with struts radiating out from it and a ball at the end of each strut. Some actually look like trees (with a ball at the end of each branch and root); that is why mathematicians use the word *tree* for this idea.

Q1 Describe the trees for which $n = 0$, for $n = 1$, and for $n = 2$.

In each of those cases, the trees are topologically identical. In other words, they are identical from the point of view of the numbers of balls and struts and of how those balls and struts are connected with each other.

Q2 Find all topologically distinct trees for $n = 3$ and for $n = 4$.

Q3 Is a polygon a tree? Explain.

A *loop* contains a strut, connected to a ball, connected to a strut, connected to a ball, and so on until the path is back to where it started. Any polygon is a loop.

Q4 Can you make a tree that contains a loop?

Q5 Can you make a tree that has a strut's end hanging free without a ball on it?

It can be proved that for a connected structure, the absence of loops and the presence of a ball at the end of each strut is equivalent to the definition of a tree.

Q6 If we took an icosahedron and removed edges from it so that what remains is a tree, how many struts would be left? (You have already done this for the Challenge. If you did not get a tree, you probably can remove more edges!)

Q7 If we took a dodecahedron and removed edges from it so that what remains is a tree, how many struts would be left? You don't need to build it to find the answer.

Now you will make a model of two *interdigitating* trees, one in the icosahedron and one in its dual dodecahedron. (*Interdigitating* means "fingers going through each other" like when you clasp your hands together.) First, recall the compound of the icosahedron and dodecahedron, which illustrated duality. Each edge of one crosses an edge of the other. There are 30 crossing points where an edge of one polyhedron crosses an edge of the other. Select either one edge or the other at each crossing, keeping one and removing one. The way to do it is to select a tree from the icosahedron's edges, as you did in the Challenge. If an icosahedron edge was removed to make the tree, then you still have the dodecahedron edge at that crossing point. But if the icosahedron edge is part of the tree, you will keep it, so you remove the dodecahedron edge it crosses.

2. To make a model of this, create a starburst of 12 r_2s and 20 y_2s in a central zomeball, which will act as scaffolding to hold it all together. Put zomeballs at the ends of all the red and yellow struts. Create any tree using icosahedron edges by inserting 11 b_2s into balls on the ends of red struts. (Either copy your tree from earlier or create a different tree.) Look to be sure there are no loops. Now, at the remaining 19 crossing points, place a b_1 between balls on the ends of yellow struts. No two blue struts should cross.

   Q8 Do the dodecahedron struts form a tree?

But how do you know this always works, no matter what tree you choose in the icosahedron? The reason the dodecahedron edges cannot have any loops is that a loop would cut off an island of remaining icosahedron parts (inside the loop and outside the loop), and you chose the icosahedron parts to form a tree; so they have no islands. The reason the dodecahedron parts are not disconnected into any islands is that disconnection would happen only if there were a loop in the icosahedron parts, and you chose the icosahedron parts to form a tree; so they have no loops.

Summarizing, the effect of this construction is to find two trees simultaneously: one in the icosahedron and one in its dual, the dodecahedron. No vertices were removed, and the total number of struts in the two trees together equals 30, the number of original edges in the two polyhedra. From this to Euler's theorem is just addition.

Let V_I be the number of icosahedron vertices, and let E_I be the number of edges in the icosahedron tree. Because it is a tree, $V_I = E_I + 1$. Let V_D be the number of dodecahedron vertices, and let E_D be the number of edges in the dodecahedron tree. Because it is a tree,

$V_D = E_D + 1$. Summing the two equations, $V_I + V_D = E_I + E_D + 2$. Now the number of dodecahedron vertices, V_D, is the same as the number of icosahedron faces, F. (This is a key aspect of duality. In the model, you'll see there is a dodecahedron vertex in the middle of each icosahedron face, establishing the one-to-one relationship.) On the right side of the equation, you can replace $E_I + E_D$ with E, the number of icosahedron edges. These substitutions give $V + F = E + 2$, which is what you wanted to show.

Although our model uses the icosahedron and the dodecahedron, the same proof works in general by creating interdigitating trees in any polyhedron and its dual.

Q9 Use a Schlegel diagram of another polyhedron to work through this proof again. Inside each face of the polyhedron, use a different color of dot to represent the vertex of the dual polyhedron (not forgetting that the outside region is a face!). Now use a thicker pen to make a tree connecting the original vertices by following some of the original edges. Finally, make an interdigitating tree connecting the colored dots. Verify that the total number of edges in the two trees equals the number of edges in the original polyhedron, and that all faces and vertices are accounted for. Explain how Euler's formula follows.

For this argument to be valid, it is necessary for the polyhedron to have a dual. As you can see on its Schlegel diagram, any simply connected polyhedron does indeed have a dual. The argument fails if the polyhedron has a hole (as in a torus—a donut shape), because after making the first tree in the construction the second structure is not a tree, because it has a loop.

24.2 Proof of Descartes' Theorem

Challenge

Prove that for polyhedra whose faces are all triangles, Descartes' theorem follows from Euler's theorem. (Hint: Sum all the angles of the polyhedron's faces in two different ways—one based on faces and one based on vertices—and make an equation of the two sums. Then do some algebra involving your equation and Euler's formula to show that the sum of the angular deficits for the polyhedron is 720 degrees.)

Descartes' theorem can be proven from Euler's theorem. The following questions will help you do this for polyhedra whose faces are all triangles.

Q1 For polyhedra composed only of triangles, derive a simple relationship between E and F. What is it?

Q2 How many vertices are in an all-triangle polyhedron with F faces? (You can substitute to eliminate E from Euler's theorem.)

Q3 What do you get if you add all the angles of all the faces of an all-triangle polyhedron assuming F faces?

Q4 What do you get if you add all the angles at each vertex, assuming V vertices and a total deficit of D?

Q5 Combine the results of Questions 3 and 4 into an equation.

Q6 Solve the system of equations from Questions 2 and 5 for D.

The following paragraph explains how to generalize this result to all polyhedra.

Start with any simply connected polyhedron, which now may have some pentagonal faces, for example. Drawing two diagonals in a pentagon face results in a new polyhedron with three triangles replacing that pentagon. Of course, Euler's theorem still holds, since it holds for any simply connected polyhedron. (Drawing a face's diagonal just adds one to both E and F, maintaining the equality.) Generally, starting with any polyhedron, draw as many diagonals as are needed to triangulate it fully. This means that every one of its faces is a triangle. This does not change the sum of the angles at any vertex because the diagonals just chop the face angles into smaller pieces that still add up to the same total. Since the face totals are unchanged by triangulation, the individual angle deficits are unchanged and the total angular deficit is unchanged. Because the triangulated polyhedron has an angular deficit equal to 720 degrees, the original polyhedron, whatever its faces, must have started with a total angular deficit of 720 degrees.

Explorations 24

A. Another Model Make a model of interdigitating trees for some other polyhedron and its dual. (The rhombic triacontahedron and icosidodecahedron are one nice possibility. The cuboctahedron and rhombic dodecahedron are another.) Review the proof of Euler's theorem on that model.

B. Descartes' Donut Euler's theorem and our proof of it assume that the polyhedra are simply connected. If we make a polyhedron that looks something like a polyhedral torus—that is, with a hole through it—then the theorem doesn't hold. In Unit 10, you found that $V + F \neq E + 2$; in fact, $V + F = E$ for this kind of polyhedron. Using that as the starting point in a proof like the above, what will the total angular deficit of a polyhedral torus be?

C. Fancy Donuts Build a polyhedral torus with five-fold symmetry for it and its hole.

D. Two-Hole Tori What happens to the two theorems in the case of a two-hole torus?

UNIT 25

Further Explorations

This is a collection of explorations that did not fit in the other units.

Goal

- To give advanced students a chance to explore further

Prerequisites

This unit is aimed at students who have covered most of the material in this book. Specific prerequisites vary from exploration to exploration.

Notes

For further activities with orthoschemes (Exploration T), there is a commercially available cardboard and Velcro kit for making the cube's orthoschemes and assembling them: *Exploring Math with Root Blocks,* by Matthew A. Solit et al.

Explorations 25

A. **Largest *n*-gon** What is the largest number of sides possible on a convex planar Zome polygon

a. without green struts?

b. including green struts?

B. **Equilateral Pyramids** An equilateral pyramid has all edges the same length. As a consequence, it has equilateral triangles for its triangular faces. What equilateral pyramids are possible in general? Can they all be built with Zome components? What about equilateral prisms and antiprisms?

C. **Antiprisms with Green** The antiprisms you worked with in Unit 1 involved no green struts. What antiprisms can you make that have green bases or zigzags?

D. **Knots** With six balls and six struts, make a loop that is tied in a simple overhand knot. Find several ways to do this.

E. **Zome Triangles** To make a triangle, you need three different directions in a common plane. Make a display of every possible Zome triangle that can be made without green struts. The display should not contain any similar triangles.

F. **Wheels** Although the Zome System does not allow construction of a regular 20-gon, it does allow a good approximation of a prism on a regular 20-sided base. The 20 dihedral angles around the circumference are equal, so this makes a nice 20-sided wheel that rolls fairly smoothly along the floor. Try to discover it. (Hint: The two 20-gon "bases" are not quite planar 20-gons made of ten b_1s and ten y_1s alternating, almost in the red plane. After you discover it, try to find an analogous 12-sided wheel with 12 equal dihedral angles.)

G. **Nested Platonic Solids** Johannes Kepler tried for years to make a model of the nested Platonic solids to explain the distances between the planetary orbits. He eventually discarded the idea. Make a Zome model of the five Platonic solids such that each except the outermost has all of its vertices on the vertices or edges of the next larger enclosing polyhedron. There is only one such nesting sequence. (It's not Kepler's.) What is it?

H. **Chiral Hexecontahedron** Using a b_2 decagon to find the center of a $b_1 + b_2 + b_1$ pentagon in the Challenge of Activity 17.2 leads to a way to dissect a regular pentagon into five congruent trapezoids. The result is chiral, with five-fold rotational symmetry but not mirror symmetry. Dissect an equilateral triangle into three congruent trapezoids in a chiral manner. A dodecahedron whose 12 faces are composed of these five-fold constructions (all of the same handedness), and an icosahedron whose 20 faces are composed of these three-fold constructions, are geometrically very distinct. However, they have the same symmetry—all the rotational axes of icosahedral symmetry, but no mirror symmetry—and so are chiral. Their structures of vertices and edges are topologically identical. Each consists of 60 pentagons (which happen to be distorted geometrically into trapezoids), 12 five-way vertices, and 80 three-way vertices. Explain why they have the same connectivity, meaning that if the struts were rubber, you could stretch either structure into the shape of the other.

I. Elevated Polyhedra To *elevate* is to build a pyramid on each face of a given polyhedron. Depending on the height of the pyramids, the faces or edges may or may not continue the lines or planes of other parts of the structure. If you elevate the cube or octahedron to just the right height, pairs of triangles merge into rhombi, giving the rhombic dodecahedron. If you start with other polyhedra elevated to a height so that two triangles merge into faces, what polyhedra can you make? Here are two to try:

- **a.** the regular tetrahedron
- **b.** a truncated icosahedron in which the pentagons have edge b_2 but the other 30 edges are of length b_1 (the hexagons are uniform)

J. Concentric Solids For each polyhedron below, find a way to build a model that would consist of two copies of it at different scales, connected in such a way as to be concentric.

- **a.** cube
- **b.** regular dodecahedron
- **c.** regular icosahedron
- **d.** rhombic dodecahedron
- **e.** rhombic triacontahedron

K. Constructions in Rhombic Polyhedra In a double-size zonohedron, there are vertices at the edge midpoints, which can be connected. In particular, in the case of rhombic polyhedra, this can lead to interesting structures. Try these two in the double-size rhombic triacontahedron:

- **a.** Inscribe a rectangle in each rhombus by connecting consecutive midpoints. Remove the red struts. What's left?

b. Connect the four edge midpoints with a new ball at each face center. Remove the original edges. We call what's left a "skewball." How would you describe it?

c. Repeat (a) and (b) with other zonohedra, especially rhombic polyhedra. You should end up with interesting polygons and skewballs.

L. Cube-Symmetry Construction Set Design a construction system like the Zome System but based on the rotational symmetries of the cube (four-fold, three-fold, and two-fold axes). What does the ball look like? The Zome System has twists in the five-fold and three-fold struts but not in the two-fold. Are there twists at the center of your three types of strut?

M. Green Holes If a next-generation zomeball contained a hole for each possible green direction, how many holes would be in each ball? Design a new ball in which green struts can each go straight out of their own hole, rather than having to share a red hole.

N. Bubbles If you dip Zome structures into a pot of bubble solution, you can create interesting bubble structures. (Use a solution of 1 part dish detergent per 6 to 8 parts water.) Examine your bubbles and count how many films meet at an edge. What is the dihedral angle where several films meet? Dip and re-dip a Zome Platonic solid, trying to create a smaller Platonic solid bubble within it. When you succeed with a cube, you get a structure like the hypercube of Unit 21. The edges of the bubble's form will connect to the edges of the Zome form. For which Platonic solids is this possible?

O. Flower Ball Any three noncoplanar struts in a ball can be extended with three more of each type to make a parallelepiped. What happens if you start with the red and yellow starburst together, make each group of three adjacent struts (two reds and a yellow) into a parallelepiped, and then keep just the exterior structure?

P. Compound of 15 Golden Bricks Build a b_3 icosidodecahedron (3, 5, 3, 5). Find eight of its vertices that are at the corners of a *golden brick*—that is, a $1 \times \tau \times \tau^2$ rectangular prism. The 30 vertices of the icosidodecahedron contain 15 such golden bricks. You can make a beautiful model showing the edges of all 15 golden bricks superimposed.

Q. Twelve-Hole Toroidal Polyhedron Build a polyhedron with 12 openings that connect to a hollow center, with every face—inside and out—a square or an equilateral triangle.

1. Construct a b_1 dodecahedron.
2. Expand each face outward with an equilateral pentagonal antiprism. These are the 12 passages to the hollow center.
3. Make each exterior pentagon the center of a decagonal saucer (five squares and five triangles surrounding the pentagon, as in the rhombicosidodecahedron, but concave). The triangle edges will meet over the edges of the original dodecahedron.

Verify that every face is a regular 3-gon or 4-gon. How many faces are there? This is just one of many fascinating polyhedra described in Bonnie Stewart's *Adventures Among the Toroids.*

R. Sangaku Problem Sangaku was a traditional Japanese practice of carving geometric problems into wooden tablets that were hung under the roofs of religious buildings. One tablet, from 1798, poses the problem of placing 30 identical small spheres around a large central sphere so that every small sphere touches four small-sphere neighbors and the central sphere. How can this be done, and what is the ratio of the two sphere radii? (Hint: Find an appropriate polyhedron and center the small spheres on its vertices.)

S. Inscribing and Circumscribing Among convex polyhedra, only the Platonic solids can be both inscribed in a sphere (vertices on the sphere) and circumscribed around it (faces tangent to the sphere at their centers). Consider some of the Archimedean solids and their duals, and for each determine either why no sphere can pass through all the vertices or why no sphere can pass through all the face centers.

T. Orthoschemes An *orthoscheme* is a tetrahedron in which each face is a right triangle. A cube can be dissected into 48 congruent orthoschemes that meet at its center. Can you build this orthoscheme? There are two mirror-image forms—24 of each. The other four Platonic solids can also be dissected into congruent orthoschemes, although they cannot be built with the Zome System. How many are there in each case?

U. How Many Axes Can There Be? We have seen examples of polyhedra with no five-fold axes, with 6 five-fold axes (in the icosahedron and dodecahedron), and with 1 five-fold axis (in a pyramid, prism, and antiprism). Can there be a three-dimensional object with 2, 3, or 4 five-fold axes? How about 0, 1, 2, 3, or 4 three-fold axes?

V. Halving Describe all the ways you cut each of the Platonic solids into two congruent halves. (Hint: Mirror planes give solutions but not the only solutions.)

W. Disphenoids A *disphenoid* is a tetrahedron with four congruent faces. A disphenoid can be made in paper by starting with any acute-angled triangle. Draw lines connecting its three edge midpoints to dissect it into four smaller similar triangles. Fold on these lines and tape the edges, making a tetrahedron. Why doesn't this work with right or obtuse triangles? What Zome disphenoids can you make?

X. *k*-Equivalence Sometimes two structures involve the same vertex positions but are connected differently, such as a regular pentagon and pentagram. Since vertices are points, which are zero-dimensional, the pentagram and pentagon are *0-equivalent,* meaning equivalent in their zero-dimensional components. The icosahedron and the great dodecahedron are *1-equivalent,* because they have the same set of edges, which are one-dimensional. This implies they are also 0-equivalent, because the endpoints of the edges give the vertices. A cube is *2-equivalent* to the unbounded inside-out polyhedron formed by taking all of three-dimensional space and removing a cubical hole; both are objects with six square faces (two-dimensional components) as their boundary. There are many examples of *k*-equivalence.

  a. Name polyhedra that are *k*-equivalent to the small stellated dodecahedron, for $k = 0$ and for $k = 1$.

  b. Find two nonconvex uniform polyhedra that are 1-equivalent to the rhombicuboctahedron (3, 4, 4, 4).

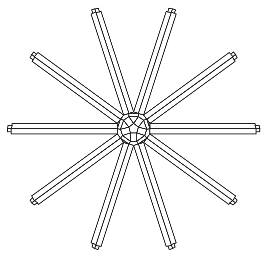
Y. Another Self-Intersecting Rhombic Triacontahedron Make a rhombus with edges of length r_4, each constructed as $r_2 + r_1 + r_2$ in order to have the proper points of intersection. Join three of these with their acute vertices meeting, to make a three-fold vertex. Add rhombi at the other acute vertices so that they are also three-fold. Continue until complete. There will be 30 rhombi in all, and the obtuse vertices meet in groups of five that go around twice. This is the dual of what nonconvex uniform polyhedron?

Z. Truncated 120-Cell Just as polyhedra can be truncated, so can higher-dimensional polytopes. Review the 120-cell of Activity 21.4. Truncating it has the effect of replacing the dodecahedra with truncated dodecahedra and creating a tetrahedron at each of the 600 vertices. You can make a three-dimensional model of this four-dimensional uniform polytope. Everywhere that the 120-cell has four dodecahedra surrounding a vertex, we will have four truncated dodecahedra with a tetrahedron nestled among them. To begin, design five progressively flattened shapes of truncated dodecahedra, analogous to the five dodecahedra used in building the 120-cell. It is helpful to build the original dodecahedra, in order to see the directions you will need. Only b_2s, r_2s, y_2s, and r_1s are needed. The truncated dodecahedra fit together exactly like the cells of the 120-cell, and tetrahedra automatically appear at the joints. Either start with the center and work out (but then you can't build the lower half) or start at the bottom and work up (using b_3s vertically as structural buttresses on balls of that height and as two vertical supports in the lowest, central, vertical 10-gon). If you have enough material to make a sizable portion of this, it may be one of the most beautiful structures you ever see.

Answers

1.1 Angles and Regular Polygons

1.



Zomeball with 10 radial blue rays

Q1 $\frac{360}{10} = 36$ degrees

2. Zomeball with 6 radial blue rays

Q2 $\frac{360}{6} = 60$ degrees

3. Zomeball with 4 radial blue rays (Red and yellow struts could also be placed in that equator; they will be discussed in Unit 13.)

Q3 $\frac{360}{4} = 90$ degrees

Q4 With an n-sided hole at the pole, there are $2n$ radial blue rays, separated by $\frac{360}{2n}$ degrees.

Q5 Zomeballs have the angles to make regular 3-, 4-, 5-, 6-, and 10-gons.

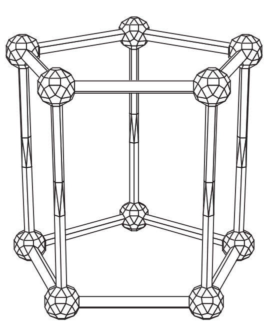
4. Regular 3-, 4-, 5-, 6-, and 10-gons

Q6

n-gon:	3	4	5	6	10
Pole hole:	3	rectangle	5	3	5

1.2 Prisms, Antiprisms, and Pyramids

1.



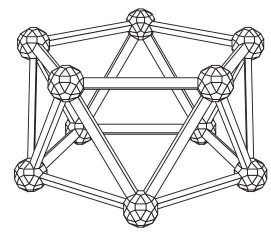
Zome pentagonal prism

2. Prisms as in 1, but with bases as 3-, 4-, 6-, and 10-gons

Q1 Cube

Q2 An n-gonal prism has $2n$ vertices, $3n$ edges, and $n + 2$ faces.

3.



Zome pentagonal antiprism

4. Four more pentagonal antiprisms:

b_2 pentagon with r_1 struts for the zigzag (or b_3 with r_2)

b_2 pentagon with b_1 struts for the zigzag (or b_3 with b_2)

b_1 pentagon with y_1 struts for the zigzag (or b_2 with y_2, or b_3 with y_3)

b_2 pentagon with y_1 struts for the zigzag (or b_3 with y_2) (This one is very short.)

5. Five triangular antiprisms:

b_1 triangles with r_1 zigzag (or all **2**s or all **3**s)

b_1 triangles with y_1 zigzag (or all **2**s or all **3**s)

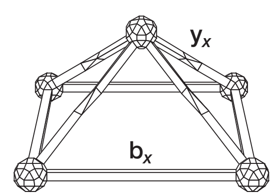
b_1 triangles with b_2 zigzag (or b_2 with b_3)

b_2 triangles with b_1 zigzag (or b_3 with b_2)

b_2 triangles with r_1 zigzag (or b_3 with r_2) (This one is very short.)

Q3 An n-gonal antiprism has $2n$ vertices, $4n$ edges, and $2n + 2$ faces.

6.



Zome square pyramid

Q4 There are five different right pyramids (a right pyramid has the apex over the center of the base). Three are constructed on one side of the base, two on the other. There are also nonright pyramids.

Q5 There are five different pentagonal right pyramids. Again, three are constructed on one side of the base, two on the other.

Q6 An n-gonal pyramid has $n + 1$ vertices, $2n$ edges, and $n + 1$ faces.

1.3 Zome System Components, Notation, and Scaling

Q1 Opposite triangles are not parallel. They are rotated 60 (or 180) degrees relative to each other. Opposite pentagons are not parallel. They are rotated 36 (or 180) degrees relative to each other.

Q2 Red and yellow struts are composed of a prism, an antiprism, and another prism, all in a stack. Because of the antiprism, the polygons at opposite ends of the strut do not have sides parallel.

Q3 The twist in the strut compensates for the twist in the ball. If the red and yellow struts did not have a twist, the balls they connect would not be parallel. Blue struts do not need a twist.

1. The possibilities are listed in answers 3 and 4 of Activity 1.2.

Q4 Regular 10-gon

2. $2b_1$ pentagonal antiprism with b_1 10-gon halfway between bases

3. $2b_1$ triangular antiprism with b_1 6-gon halfway between bases

Q5 Regular $2n$-gon

Q6 Zomeballs have the angles to make only regular 3-, 4-, 5-, 6-, and 10-gons. So, the only values of n for which both regular n-gons and regular $2n$-gons can be built are 3 and 5.

Explorations 1

A. There are twelve Zome red, yellow, and blue skew polygons: five skew 10-gons that come from the zigzags of the 5-gonal antiprisms, five skew 6-gons from the triangular antiprisms, the skew 6-gon of the cube, and one more yellow 6-gon. (There are also skew polygons that can be made with green. An exploration in Unit 14 shows how to find them all.)

B. Start with any polygon base, add any strut in any hole (other than the pole) of any ball, place an identical parallel strut in each of the other balls, and connect the top of the new struts to make a parallel polygon.

C. An equiangular $2n$-gon with every other side twice the length of the others.

D. Blue rhombi can have a smaller angle of 36, 60, 72, or 90 and a supplementary larger angle. All seven rhombi can be raised into pyramids. Try b_2 for the base. For the red and yellow rhombi and the square, the apex of the pyramid can be directly above the center. For the other three blue rhombic pyramids, the apex will be off-center. All blue edges can be used in each blue case, even for the square if you put the apex outside the box. Students can continue to explore.

E. It is a kind of concave pentagonal antiprism consisting of two regular 5-gons and ten isosceles triangles. It is indented along the b_2 edges. Imagine an ordinary antiprism with a rubber zigzag that was given an extra large twist. We often assume polyhedra are convex and ignore concave cases like this.

2.1 Building and Counting

Q1 The icosahedron has 20 faces, 30 edges, and 12 vertices.

Q2 The icosahedron has 3 sides on each face and 5 edges meeting at each vertex.

Q3 There are six antiprisms in an icosahedron. Hold the icosahedron by any pair of opposite vertices; there is an antiprism halfway between your hands. The 12 vertices come in 6 pairs.

Q4 The dodecahedron has 12 faces, 30 edges, and 20 vertices.

Q5 The dodecahedron has 5 sides on each face and 3 edges meeting at each vertex.

Q6 The same numbers appear, but with different roles. Since the icosahedron has 3 sides on each of 20 faces and the dodecahedron has 5 sides on each of 12 faces, each has 30 edges. For each vertex of the dodecahedron, there is an icosahedron edge and vice versa. This will be explored further in Unit 9, Duality.

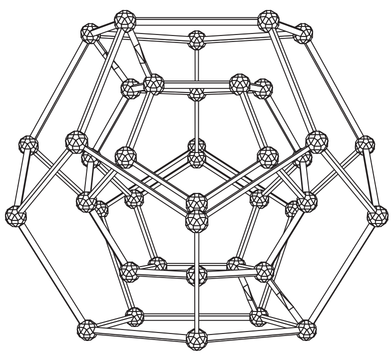
2.2 Scaling

Challenge They are equal. See the answer to Question 3.

Q1 Pentagon *B* is twice the size of pentagon *A* in all corresponding lengths. A b_1 pentagon has a b_2 diagonal, so a $2b_1$ pentagon has a $2b_2$ diagonal.

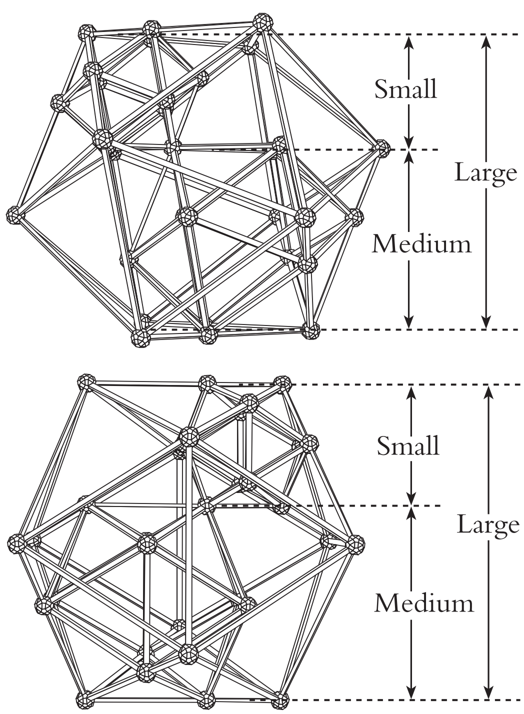
Q2 2 : 1

3.



Zome concentric dodecahedra

4.



Zome scaled icosahedra

Q3 If you place the model on the table resting on a face of the medium icosahedron, a face of the small icosahedron will be at the top. Looking from the side and imagining a ruler standing vertically, you see that the distance between the opposite faces of the medium icosahedron plus the distance between opposite faces of the small icosahedron gives the distance between opposite faces of the large icosahedron. Similarly, resting it on a vertex or an edge shows the other two relationships.

5. Again, small plus medium equals large.

Explorations 2

A. One strategy is to maintain five-fold symmetry, using the ten blues as two regular pentagons, which are the bases of a pentagonal antiprism with either a red or a yellow skew 10-gon zigzag. The remaining ten struts (yellow or red) are used to make each end into a pentagonal pyramid. There are several variants of this solution with different choices for the lengths.

B. Five-fold symmetry is again natural. One nearly flat dodecahedron has ten b_2s used for regular pentagons in the top and bottom face. An almost flat y_2 skew decagon is the zigzag equator, and r_1s join those components. It is essential to check that the ten side pentagons (each b_2–r_1–y_2–y_2–r_1) really are planar. (They are.) Another variant is very tall with b_1 pentagons, an r_2 skew 10-gon equator, and y_3s connecting. The ten side pentagons are each b_1–y_3–r_2–r_2–y_3.

C. The elevated dodecahedron has 60 faces, 32 vertices, and 90 edges.

D. The concave equilateral deltahedron also has 60 faces, 32 vertices, and 90 edges. It is topologically equivalent to the elevated dodecahedron.

E.–F. The rhombic triacontahedron has 30 red rhombic faces, 32 vertices, and 60 edges.

3.1 Green Polygons

Challenge The equilateral square pyramid is half of the regular octahedron.

Q1 The single green struts in the six faces of the cube form a regular hexagon.

Q2 Only the square and the octagon. Use b_1 vertical edges.

3.2 The Regular Tetrahedron and Octahedron

Q1 The regular octahedron is an equilateral triangular antiprism. Any of its four pairs of opposite sides can be considered the bases of the antiprism. It is also a *square dipyramid*—two square pyramids joined at their bases.

Q2 The edges of the regular octahedron form three mutually perpendicular squares, each an "equator" when two opposite vertices are considered poles.

Q3 A regular octahedron

7. In a double-scale tetrahedron, insert just one of the three squares of the octahedron at its core. You can rest the tetrahedron on an edge to make this cross section horizontal.

3.3 Only Five Platonic Solids: A Proof

Challenge The pentagonal antiprism with equilateral sides satisfies just the first and third conditions. Gluing two pentagonal pyramids together 5-gon to 5-gon would make a *dipyramid* of ten equilateral triangles, which satisfies just the first and second conditions, but it is not Zome-constructible. You can make a tetrahedron in which each face is an isosceles triangle made of a $\mathbf{b_2}$ and two $\mathbf{y_2}$s; it satisfies just the second and third conditions.

Q1 *See table at bottom of page.*

Q2 In the icosahedron/dodecahedron pair, the entries in the first two columns (5 and 3) switch places, as do the entries in the next two columns (20 and 12), and the entry in the last column is identical (30). The cube and octahedron are pairs in exactly the same way. The tetrahedron does not have another polyhedron to pair with, but it pairs with itself.

Q3 Every polygon has at least three sides; two sides cannot enclose any area.

Q4 Every vertex has at least three faces meeting; two would not enclose any volume.

Q5 $\{3, 6\}$ is flat because every vertex has 360 degrees.

Q6 $\{3, 7\}$, $\{3, 8\}$, and so on, have more than 360 degrees at each vertex.

Q7–Q9 $\{4, 4\}$ and $\{6, 3\}$ are flat. Everything else has more than 360 degrees at each vertex.

Q10 All possibilities are accounted for in Questions 3–9. The five possibilities are the Platonic solids.

3.4 Truncation

Challenge The truncated cube consists of six 8-gons and eight 3-gons. Truncating a $\mathbf{3b_1}$ model gives irregular octagons, so rebuild with **gb** struts.

Q1 A square

Q2 A hexagon

Q3 Truncated tetrahedron: four 6-gons, four 3-gons. Truncated octahedron: six 4-gons, eight 6-gons. Truncated cube: six 8-gons, eight 3-gons. Truncated icosahedron: twenty 6-gons, twelve 5-gons. Truncated dodecahedron: twelve 10-gons, twenty 3-gons.

Q4 $F + V$

Q5 An octahedron

Polyhedron	Sides on each face	Faces at each vertex	Faces	Vertices	Edges
tetrahedron	3	3	4	4	6
octahedron	3	4	8	6	12
cube	4	3	6	8	12
icosahedron	3	5	20	12	30
dodecahedron	5	3	12	20	30

Explorations 3

A. Like the cube, the octahedron has a regular 6-gon cross section. Rest it on a face and imagine slicing it horizontally through the center. The icosahedron and the dodecahedron both have regular 10-gon cross sections. Hold a five-fold axis vertically and slice horizontally through the center. The regular tetrahedron has a square cross section. Rest it on an edge and imagine slicing it horizontally through the center.

B. The 12 octahedron edges come in 6 opposite pairs, each equal to and parallel to one of the original tetrahedron's 6 edges.

C. Choose four sides of the octahedron, no two adjacent. Imagine the planes of those faces extending off into space. The region they enclose is a large tetrahedron, divided into five parts: four tetrahedra surrounding the original octahedron core. Mathematically the large tetrahedron exists, but lines where the planes intersect may not be Zome-constructible.

D. This will also work starting from any octahedron that has opposite faces congruent.

E. The formula $y\left(180 - \frac{360}{x}\right)$ gives the sum of the interior angles when y x-gons meet at a vertex. This must be less than 360 to make a convex polyhedron. The plot is symmetric about the line $x = y$. The paired polyhedra—icosahedron/dodecahedron, cube/octahedron—are at mirror-image positions. The tetrahedron is on this line, mirrored to itself.

F. These can all be built. Start with triple-scale models for the truncated polyhedra or with double-scale models for the ones truncated to the edge midpoint. When truncating to the edge midpoints, the octahedron and cube give the same result (the cuboctahedron); the icosahedron and dodecahedron also give the same result (the icosidodecahedron).

G. The cube, octahedron, and icosahedron contain regular skew hexagons. Arrange one with a three-fold axis vertical and look around the "equator" halfway up. (With the dodecahedron, one finds six edges that can be extended to meet, forming a regular skew hexagon.) The regular octahedron contains four regular skew 6-gons—the zigzags when you look at the octahedron as an antiprism in four different ways. There are three regular skew 4-gons in the regular tetrahedron—the four edges that remain after removing any two opposite edges.

4.1 Counting Strategies

Q1 Resting the icosahedron on a vertex, notice 5 edges meeting at the top, 5 in a pentagon around it, 10 in the zigzag, 5 in the lower pentagon, and 5 touching the table. 5 + 5 + 10 + 5 + 5 gives 30 edges total. The 10 can be seen as 5 + 5 because 5 edges slope uphill and 5 slope downhill as you walk around the icosahedron.

Q2 Resting the dodecahedron on a face, there are 5 edges in the top pentagon, 5 radiating from it, 10 in the zigzag equator, 5 radiating from the bottom, and 5 in the bottom face, making 30 total.

Q3 Four edges on the top face, four on the bottom face, and four verticals total to 12.

Q4 Three edges touching the top vertex, three touching the table, and six in the zigzag equator (a skew hexagon) total to 12.

Q5 The 20 faces of the icosahedron contribute 3 edges each, making 60. However, each edge would be counted twice as it is part of two triangles. So, dividing 60 by 2 gives 30 edges.

Q6 The dodecahedron has 12 faces of 5 vertices each. Multiplying 12 by 5 equals 60 gives each vertex overcounted by a factor of 3, as it is part of 3 faces. Correcting for this, there are 60 divided by 3 equals 20 vertices. Similarly, there are $\frac{12(5)}{2} = 30$ edges.

Q7 The octahedron has $\frac{8(3)}{4} = 6$ vertices and $\frac{8(3)}{2} = 12$ edges.

Q8 A polyhedron with n faces of k sides each has $\frac{n(k)}{2}$ edges. For the cube, $\frac{6(4)}{2} = 12$ edges.

Q9 They must have counted wrong or not realized that some face is not triangular, because by the method of Question 8, the number of edges would not be an integer.

Q10 A polyhedron with n faces of k sides each, with d faces meeting at each vertex, has $\frac{n(k)}{d}$ vertices. For the cube, $\frac{6(4)}{3} = 8$ vertices.

Q11 $\frac{n_1k_1 + n_2k_2}{2}$ edges

Q12 $\frac{vd}{2}$ edges, because each edge touches two vertices

Explorations 4

A. The 60-sided concave equilateral deltahedron is analogous (and its framework of edges is topologically equivalent) to the elevated dodecahedron described in the problem. For the rhombic triacontahedron, the fact that there are thirty 4-sided faces implies that there are 60 edges, by the formula of Question 8.

B. This is not possible. There must be an even number of faces with an odd number of sides. Let $a, b, c, \ldots, n$ be a list in any order of the number of sides on each face; for example, for a cube it is 4, 4, 4, 4, 4, 4. Then the number of edges is $\frac{(a + b + c + \cdots + n)}{2}$. The numerator must be even; otherwise, the number of edges would not be an integer. So the number of odd numbers summed must be even. Question 8 is also related to this result.

C. No. The number of people who shake hands an odd number of times is even. This is analogous if you think of each person as a face of a polyhedron and a handshake as corresponding to two faces sharing an edge.

D. The sculpture contains 10 in each of 12 pentagons, plus 30 along the hexagon edges that connect pairs of pentagons, totaling 150.

5.1 Icosahedron and Dodecahedron Symmetries

Challenge This is the number of mirror planes. For the icosahedron (and the dodecahedron), it is 15. For the cube and octahedron, it is 9. For the tetrahedron, it is 6.

Q1 6, 10, 15. This is half the number of vertices, faces, and edges, respectively.

Q2 $kn = 30$

Q3 By the definition, this is true, but it is not interesting, so these axes are not usually counted.

Q4 There is one mirror plane for each pair of opposite edges, 15 planes in all.

Q5 The five-fold axes pass through the centers of opposite faces. The three-fold axes pass through the centers of opposite vertices. The two-fold axes pass through the midpoints of opposite edges. Each mirror plane contains a pair of opposite edges. Compared to the icosahedron, the roles of faces and vertices are switched.

Q6 6, 10, 15, 15

Q7 There are the same numbers of corresponding symmetry elements.

4. The model is a zomeball with every hole filled in using a nongreen strut. The directions of the holes in the zomeball were chosen as the directions of the icosahedral symmetry axes.

5.2 Simple Polyhedra Symmetries

Challenge Cut four of the sides (left, right, front, and back) with green diagonals in a way that every vertex touches one diagonal. There is still a vertical four-fold axis, and there are 4 two-fold axes in the horizontal plane—two axes go through the midpoints of opposite green struts, and two go through the midpoints of opposite vertical blue edges.

Q1 There are 5 two-fold axes. Each goes through the midpoint of a vertical edge and through the center of the opposite rectangular face. There are 6 mirror planes. If the prism rests with a base on the table, one mirror is horizontal, halfway between the top and bottom. Each of the other five mirrors contains a vertical edge and goes through the center of the opposite rectangle. The axes are contained in the mirror planes.

Q2 A right prism on a regular n-gon base has one n-fold axis perpendicular to the bases and n two-fold axes in the plane halfway between its bases. For n odd, each two-fold axis goes through one edge and the center of the opposite face. For n even, half the two-fold axes connect opposite edge midpoints and the other half connect opposite face centers. In either case, there are $n + 1$ mirrors, including the plane halfway between the bases.

Q3 There are 5 two-fold axes. Each goes through the midpoints of two opposite zigzag edges. There are only five mirror planes. (Not six; the plane midway between the bases is not a mirror plane.) Each of the five is vertical (meaning it contains the five-fold axis) and contains two opposite vertices. It splits two opposite triangles. Another difference between the prism and antiprism symmetry is that, in the antiprism, the two-fold axes are not contained in the mirror planes; they are halfway between the mirror planes.

Q4 The pentagonal pyramid has 1 five-fold axis and no other axes. There are five mirror planes. Each contains the five-fold axis and one slanting edge, and splits the opposite triangle in half. The square pyramid has 1 four-fold axis and no other axes. There are four mirror planes. Two contain the axis and two slanting edges, and two split two triangles in half.

Q5 The brick has 3 two-fold axes (each connecting the center of two opposite faces) and three mirror planes (parallel to and halfway between opposite faces).

Q6 The result has the five-fold axis and the 5 two-fold axes of the original prism, but no mirror planes.

7. In one, the top saw blade goes clockwise; in the other, the bottom one does.

5.3 Cube and Related Symmetries

Challenge See the answer to Question 3.

Q1 The cube has 3 four-fold axes, which connect the centers of opposite faces. There are 4 three-fold axes, which connect opposite vertices. There are 6 two-fold axes, which connect the midpoints of opposite edges. There are two different kinds of mirror planes. Three mirror planes are halfway between opposite pairs of faces, and six mirror planes each contain a pair of opposite edges. There are nine mirror planes total.

Q2 The octahedron has 3 four-fold axes, which connect opposite vertices. There are 4 three-fold axes, which connect the centers of opposite faces. There are 6 two-fold axes, which connect the midpoints of opposite edges. There are two different kinds of mirror planes. Three mirror planes are halfway between opposite pairs of vertices, and six mirror planes are halfway between pairs of opposite edges. There are nine mirror planes total.

Q3 The cube and octahedron have exactly the same symmetry. If one sees past their geometric forms to the underlying axes and mirrors, they are identical.

Q4 The tetrahedron has 4 three-fold axes (each connecting a vertex with the center of the opposite face), 3 two-fold axes (each connecting the midpoints of opposite edges), and 6 mirror planes (each containing one edge and the opposite edge's midpoint).

Q5 The cube and tetrahedron have their three-fold axes in the same directions. The two-fold axes of the tetrahedron align with the four-fold axes of the cube. (The two-fold axes of the cube are unrelated to the axes of the tetrahedron.) The mirror planes of the tetrahedron are six of the mirror planes of the cube. (The remaining three mirrors of the cube are unrelated to the mirrors of the tetrahedron.)

Q6 The pyritohedron has 3 two-fold axes, 4 three-fold axes, and 3 mirror planes. It is different from the cube's symmetry because there are two-fold axes where the cube has four-fold axes. It has 3 mirror planes, parallel to the cube's faces. These are the three mirrors of the cube's nine that are not mirrors in a tetrahedron.

Q7 The pyritohedron has exactly the same rotational axes as the regular tetrahedron but different mirror planes.

Q8 The Zome cube, when strut orientation is taken into account, has pyritohedron symmetry.

Explorations 5

A. A blue strut extending each pentagon edge in the dodecahedron happens to lie on a mirror plane and does not create a chiral object. One option is to use a yellow strut to make a spiral inside each face.

B. Cut through the center with a plane perpendicular to any three-fold axis. In a **2**$\mathbf{b_1}$ dodecahedron, the hexagon has edge $\mathbf{b_3}$. In a **2**$\mathbf{b_2}$ dodecahedron, all ten hexagonal slices can be shown at once.

C. That 16-hedron has the rotational symmetry of a regular tetrahedron, but no mirror planes. There are

4 three-fold axes. (Each passes through the middle of a triangle to the opposite vertex.) There are 3 two-fold axes. (Where a b_1 is opposite another b_1, a two-fold axis connects their midpoints.) The relation to the tetrahedron may be clearer if you remove all the b_2s, to get a tetrahedral form in which each of the six edges consists of three b_1s in a Z shape.

D. These have the same symmetry as the pyritohedron, 3 two-fold axes, 4 three-fold axes, and 3 mirror planes.

E. This also has the same symmetry as the pyritohedron.

6.1 Faces, Vertices, and Edges

Challenge See the answer to Question 1.

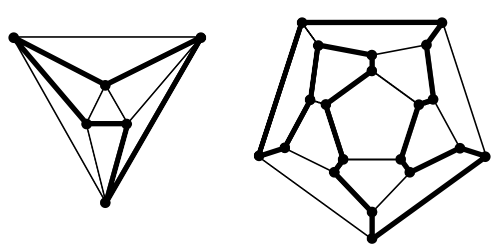
1. *See table in second column.*

Q1 Euler's theorem: $V + F = E + 2$ (or another form of this equation, such as $V - E + F = 2$). It holds for any convex polyhedron.

Q2 Substituting $F = n$ and $E = \frac{3n}{2}$ into $V + F = E + 2$ gives $V + n = \frac{3n}{2} + 2$.
So, $V = \frac{n}{2} + 2$.

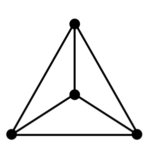
6.2 Topology

Challenge Use the Schlegel diagrams of an octahedron and a dodecahedron to find a round-trip path.



Octahedron and dodecahedron with round-trip paths marked

Q1 a.

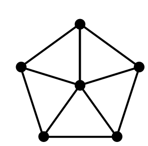


Tetrahedron

6.1, 1. *(from previous column)*

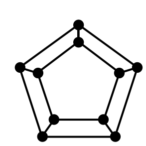
Polyhedron	***F***	***V***	***E***
tetrahedron	4	4	6
regular octahedron	8	6	12
cube	6	8	12
regular icosahedron	20	12	30
regular dodecahedron	12	20	30
triangular prism	5	6	9
pentagonal prism	7	10	15
n-gon prism	$n + 2$	$2n$	$3n$
triangular antiprism	8	6	12
pentagonal antiprism	12	10	20
n-gon antiprism	$2n + 2$	$2n$	$4n$
square pyramid	5	5	8
pentagonal pyramid	6	6	10
n-gon pyramid	$n + 1$	$n + 1$	$2n$

b.



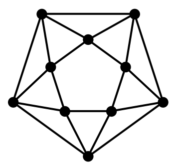
Pentagonal pyramid

Q2 a.



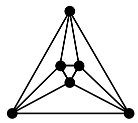
Pentagonal prism

b.



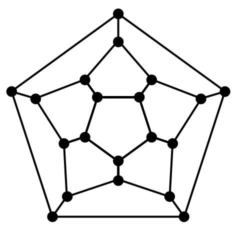
Pentagonal antiprism

Q3 **a.**



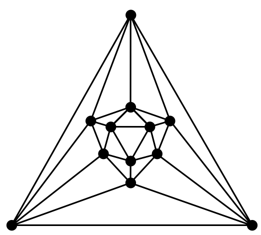
Octahedron

b.



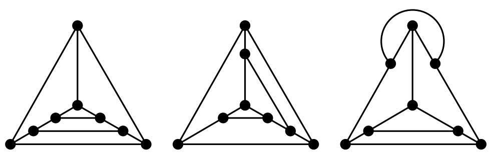
Dodecahedron

c.

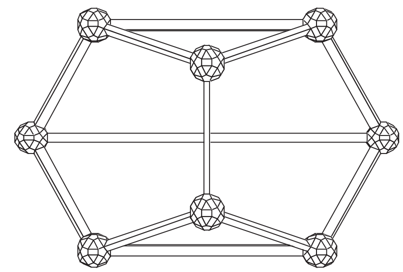


Icosahedron

Q4 One method is to start with a tetrahedron ($V = 4$, $E = 6$) and add two new edges, each to two new vertices. The figure shows three ways of drawing this, but they all correspond to the same polyhedron topologically. It has two 3-gons, two 4-gons, and two 5-gons. There are many ways to construct a geometric realization of it, as you can choose the angles and lengths. One choice, easy to build with the Zome System, is to start with two regular $\mathbf{b_1}$ pentagons joined as in a dodecahedron, then add two $\mathbf{b_2}$s and a $\mathbf{b_3}$ so that the triangles will be isosceles and the 4-gons will be trapezoids.



Three Schlegel diagrams of polyhedra with 8 vertices and 12 edges



Zome noncube polyhedron with 8 vertices and 12 edges

Q5 Any convex polyhedron with 8 vertices and 12 edges will have 6 faces.

Q6–Q7 When a pyramid is erected over an n-gon face, that face is replaced by n triangular faces, and a new vertex and n edges are added. So, F increases by $n - 1$, V increases by 1, and E increases by n. Both sides of $V + F = E + 2$ increase by n, preserving the equality.

Q8–Q9 Truncating a vertex where n faces meet eliminates the vertex but adds n new vertices and n new segments surrounding the new face. So, V is increased by $n - 1$, F is increased by 1, and E is increased by n, again preserving equality.

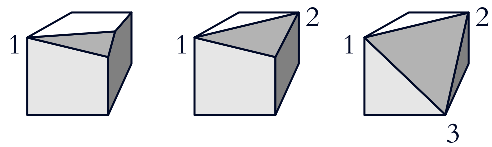
Explorations 6

A. Euler's formula, $V + F = E + 2$, also applies to figures in the plane. The example shows that Euler's theorem applies more generally than just to polyhedra.

B. There are five altogether, including the 5-gon prism. If you truncate one vertex of a cube, you get a polyhedron with $6 + 1 = 7$ faces and 10 vertices. (It has three 4-gons, three 5-gons, and one 3-gon.) Alternatively, you can truncate three vertices of a tetrahedron to get a polyhedron with $4 + 3 = 7$ faces and 10 vertices. (It has three 3-gons, three 5-gons, and one 6-gon.) Two others result from truncating two vertices of a triangular prism. In light of Euler's theorem, we do not need to count the edges of these different models, because any convex polyhedron with 7 faces and 10 vertices must have 15 edges.

C. A pyramid on a hexagonal base has $V = 7$ and $F = 7$. Cutting off one corner of a cube with a deeper cut, so the slicing plane touches one, two, or three adjacent vertices, gives polyhedra with $F = 7$

and $V = 9, 8$, or 7, respectively. To get $V = 6$, do this last operation on both sides, leaving a square base and six triangular faces.



Three cubes with one corner truncated, giving polyhedra with $F = 7$ and $V = (9, 8, 7)$

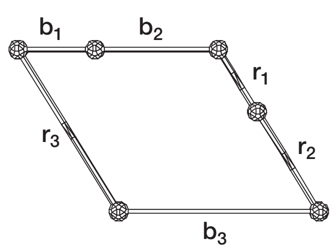
D. The missing part of the proof is that $E \geq \frac{3}{2}V$, which is dual to $E \geq \frac{3}{2}F$. One way to see it directly is to imagine snipping each strut in half at its midpoint. Then each vertex is connected to at least three half-edges. As the number of vertices must be at least 4, $E = 7$ implies $V = 4$. But by Euler's theorem, $F = 4$ and $V = 4$ implies $E = 6$ (giving the tetrahedron), contradicting the assumption that $E = 7$. This result and proof were first given by Euler himself.

E. For $(p - 2)(q - 2) < 4$, the only possibilities for $(p - 2)$ and $(q - 2)$ are 1, 1 or 1, 2 or 2, 1 or 1, 3 or 3, 1. These give us the Platonic solids $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$, and $\{5, 3\}$, respectively: the tetrahedron, octahedron, cube, icosahedron, and dodecahedron.

F. From $2 = k\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right)$ and $qV = k$, we eliminate k and solve for V to get $V = \frac{4p}{2p - pq + 2q}$. And from $pF = k$ we get $F = \frac{4q}{2p - pq + 2q}$. Notice the duality: If we change p to q and vice versa, the formulas for V and F are interchanged.

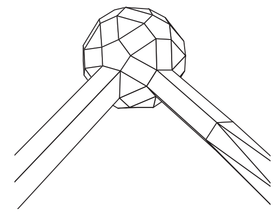
7.1 Finding the Patterns Using Geometry

1. Answers will vary, depending on the parallelograms, but here is an example:



Zome parallelogram of six struts

Q1–Q3 Opposite angles of the figure are equal, as shown by inspecting the balls at the vertices.



Zomeball with red and blue struts in acute angles of previous figure

Therefore, this is a parallelogram.

Q4 Opposite sides of a parallelogram are equal.

Q5 Students will find that not every isosceles triangle they can make using the sum property will allow the extra zomeball(s) to be connected as the problem requires. A triangle with a $\mathbf{b_2}$ base and one equal side of $\mathbf{b_1} + \mathbf{b_2}$ will work if the $\mathbf{b_1}$ strut is adjacent to the base. The smallest isosceles triangle has the same vertex angle as the largest one, as shown by inspecting the zomeballs at the vertices. Or you can use the fact that the isosceles triangles share one base angle, so all the corresponding angles must be equal, and therefore the triangles are similar.

Q6 Since the triangles are similar, their corresponding sides are proportional, which proves the ratio pattern.

Q7 Students will find that not every isosceles triangle they can make using the sum property will allow the extra zomeball(s) to be connected as the problem requires. The equal side made up of $\mathbf{y_1} + \mathbf{y_2}$ and the base of $\mathbf{b_1} + \mathbf{b_2}$ must have the shorter part of the side next to the same vertex. The smallest isosceles triangle has the same vertex angle as the largest one, as shown by inspecting the zomeballs at the vertices. Or you can use the fact that the isosceles triangles share one base angle, so all the corresponding angles must be equal, and therefore the triangles are similar.

Q8–Q9 $\frac{\mathbf{b_2}}{\mathbf{y_2}} = \frac{\mathbf{b_3}}{\mathbf{y_1} + \mathbf{y_2}}$; therefore $\frac{\mathbf{b_2}}{\mathbf{y_2}} = \frac{\mathbf{b_3}}{\mathbf{y_3}}$, which is equivalent to $\frac{\mathbf{y_3}}{\mathbf{y_2}} = \frac{\mathbf{b_3}}{\mathbf{b_2}}$. A similar proof shows the other equalities.

7.2 Finding the Patterns Using Measurement

Challenge They would measure to make sure the lengths sum properly and are in the proper ratio.

Q1 Answers should be within a millimeter of these measurements:

Strut	Length
b_1	75
b_2	121
b_3	196
y_1	65
y_2	105
y_3	170
r_1	71
r_2	115
r_3	186

Q2 Sums should be within a couple millimeters of 196, 170, 186.

Q3–Q4 Answers will be near 1.62.

Q5 1.62

Q6 $b_0 = 46$ mm, $y_0 = 40$ mm, $r_0 = 44$ mm, $b_4 = 318$ mm, $y_4 = 275$ mm, $r_4 = 301$ mm

Q7 Yes, it does.

7.3 The Golden Ratio and Scaling

Challenge See the answer to Question 3.

Q1 τ, τ^2

Q2 $1 + \tau = \tau^2$ or $\tau^2 - \tau - 1 = 0$

Q3 $\tau = \frac{1 \pm \sqrt{5}}{2} \approx 1.618 \ldots$

Q4 **a.** $\frac{1}{\tau} = 0.618 \ldots$

b. $\tau^2 = 2.618 \ldots$

Q5 τ

Q6 Multiply both sides of the equation by τ, or by τ^n.

Q7 $x_4 = x_2 + x_3$

$x_5 = x_3 + x_4 = x_3 + x_2 + x_3 = x_2 + 2x_3$

1. Possibilities include y_2–b_3–r_3 and y_3–($b_2 + b_3$)–($r_2 + r_3$).

Q8 The ball is a physical object that models the vertex—a mathematical point at its center. The ball could be made larger or smaller and the strut correspondingly smaller or larger as long as the centers remain at the same distance. The actual plastic struts do not satisfy either the sum or the ratio patterns since they are each one zomeball diameter less than the values that do. Let the ball diameter be d; then ($b_1 - d$) is the short plastic strut length, one-half diameter less on each end than the center-to-center distance b_1. Summing ($b_1 - d$) and ($b_2 - d$) does not give ($b_3 - d$), as it gives ($b_3 - 2d$). This can be seen visually by laying the struts next to each other without balls on the ends.

Explorations 7

A. Annexing a square along the long edge of a golden rectangle creates a golden rectangle of the next larger size. Conversely, removing a square from a golden rectangle leaves a golden rectangle of the next smaller size.

B. The scaling factor is τ^2 because the next pentagon is size b_3. After that comes b_5, which can be built as $2b_3 + b_2$.

C. A 36-degree V of b_2s is the sides of an isosceles triangle with a b_1 base. So, scaling down by τ, a 36-degree V of b_1s places two balls at b_0 distance, even though there is no strut that fits in there. Using two of those "invisible b_0s" in a 36-degree V gives $b_{(-1)}$ separation. Repeating smaller, we see the balls touch when we create a $b_{(-2)}$ center-to-center separation.

D. A plot shows that the ratios $\frac{F_{i+1}}{F_i}$ approaches τ as i gets large. They wiggle around the value 1.61803 . . . , alternately too large or too small, but always getting closer to it.

E. The ratios of consecutive terms in any Fibonacci-like sequence approach τ. The same limiting ratio is approached no matter what integers are chosen as starting values. The sequence discussed in this unit, $1, \tau, \tau^2, \tau^3, \ldots$, is both a geometric sequence and a Fibonacci-like sequence at the same time. The only other number with this property is the negative root to the quadratic equation satisfied by τ.

8.1 Basic Tessellations

1. Whatever triangle you make will tessellate the plane. One technique for building the tessellation is to alternate the original triangle with a copy that is rotated 180 degrees, thereby creating a parallelogram that repeats to form a strip. Then the strips can be juxtaposed.

Q1 The three different interior angles sum to 180 degrees. In one possible tessellation, each angle appears twice at every vertex of the tessellation, to make 360 degrees around every vertex.

2. Whatever quadrilateral you make will tessellate the plane. One way to generate the tessellation is to rotate the quadrilateral 180 degrees around the midpoint of a side in order to get the next copy of the tile.

Q2 The four interior angles sum to 360 degrees, and each angle appears once at every vertex of the tessellation.

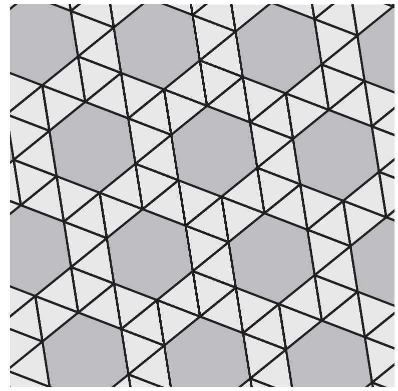
3. With the Zome System, two sides of the pentagon must be parallel. A pentagon made by joining a parallelogram and a triangle will work. Outside of the Zome System, other pentagons will tessellate. The pentagon cannot be regular, and there must be more than one type of vertex.

Q3 The sum of the angles around a vertex is 360 degrees.

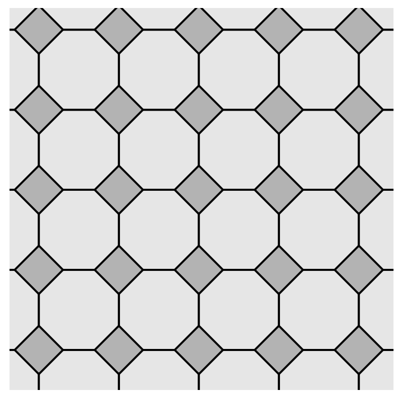
4. There are three: equilateral triangles, squares, and regular hexagons. No others are possible with just one type of regular polygon, because for *n* polygons to meet at a vertex, the interior angle of the polygon must be $^{360}/_{n}$, and these three are the only such polygons.

Q4 Answers will vary.

5. (3, 6, 3, 6) is illustrated with the question. (3, 3, 3, 3, 6) and (4, 8, 8) are illustrated here. Students might sketch, but cannot construct with the Zome System, any of (3, 12, 12), (3, 3, 3, 4, 4), (3, 3, 4, 3, 4), (3, 4, 6, 4), or (4, 6, 12).



Archimedean tessellation (3, 3, 3, 3, 6)



Archimedean tessellation (4, 8, 8)

Q5 Both polyhedra and tessellations involve polygons meeting edge-to-edge. In both cases there are special highly structured cases that are regular or Archimedean. (Archimedean polyhedra are the subject of Unit 12.) Because tessellations are planar, the sum of the vertex angles is always 360 degrees. In polyhedra, the sum of the angles at any vertex can vary; in convex polyhedra, it is always less than 360 degrees, as we will see in Unit 10.

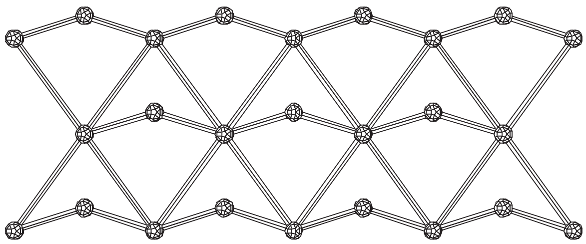
8.2 Nonperiodic Tilings

1. One solution is to use the 36-, 60-, or 72-degree rhombus, making one tessellation by simply translating the rhombus and another tessellation by reflecting the rhombi in one row to get the next row. (This will not work with red or yellow Zome System rhombi, even though it is easy to make a rhombus tessellation based just on translation.) Changing the length of just one direction of lines changes the tessellation to a repetition of parallelograms.

Q1 One solution is to start with the tessellation of regular hexagons and dissect them all into rhombi, giving a tessellation with one type of face and two types of vertices (three-fold and six-fold).

Q2 The kites have interior angles of 72, 72, 72, and 144 and the darts of 72, 36, 216, and 36.

2.

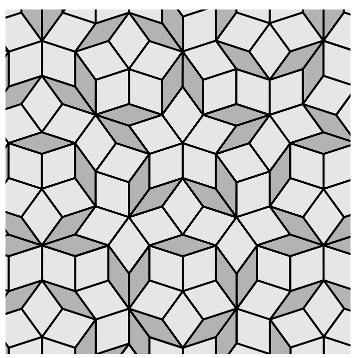


Zome tessellation of kites and darts

3. This is Zome-constructible in blue.

4. The method used in the first model of alternating the slants will work here as well.

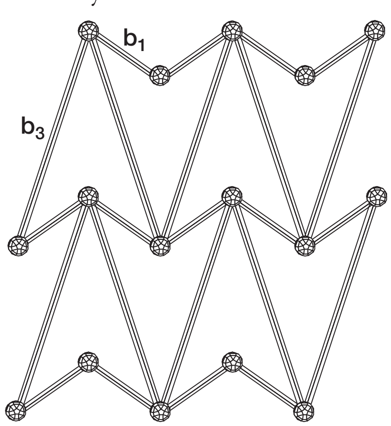
5.



A nonperiodic tiling of Penrose rhombi

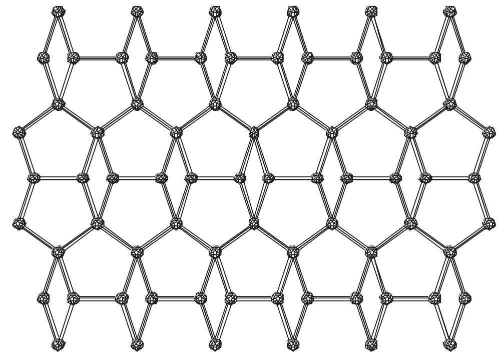
Explorations 8

A. One possibility:



Zome tessellation with a nonconvex quadrilateral

B. One possibility:



Zome tessellation with pentagons and rhombi

D. Scale so that the triangles are b_2–b_3–b_3.

F. All three are Zome-constructible in blue.

9.1 Dual Tessellations

Q1 They would form the tessellation of equilateral triangles.

Q2 They would form another tessellation of squares.

Q3 A self-dual structure is dual to a copy of itself.

Q4 The tessellation of hexagons again

Q5 A tessellation of 60-degree rhombi

Q6 The dual of the dual always gives back the original tessellation. An edge at right angles to the right angle gives the original direction again.

Q7 **a.** If the struts in the hexagonal tessellation are of length 1, then use of the Pythagorean theorem shows that each strut in the dual tessellation of triangles should have length $\sqrt{3}$. The length of the next larger size strut is 1.618, not quite 1.732. If we made a large enough model, this discrepancy would show up clearly.

b. We know $T = \sqrt{3}\,H$ in the dual tessellations, where T is the triangle strut length and H is the hexagon strut length. Therefore, $H = \frac{T}{\sqrt{3}}$ or, equivalently, $H = \left(\frac{\sqrt{3}}{3}\right)T$.

c. The two factors are reciprocal.

9.2 Dual Platonic Solids

Challenge Cube and octahedron, or icosahedron and dodecahedron, or two tetrahedra.

Q1 The octahedron. Four octahedron edges meet above each cube face.

Q2 The cube. Again, the dual to the dual gives back the original structure.

1. This can be done with a $\mathbf{2b_1}$ cube and a $\mathbf{2g_1}$ octahedron (or $\mathbf{2b_2}$ and $\mathbf{2g_2}$). Their edges will cross each other at right angles at the midpoints.

Q3 The compound of cube and octahedron in dual position shows that there is a cube vertex in the middle of each octahedron face and an octahedron vertex in the middle of each cube face. So the fact that one has 6 faces and 8 vertices while the other has 8 faces and 6 vertices is not a coincidence but a consequence of the fact that they are dual.

2. This can be done with a $\mathbf{2b_2}$ icosahedron and a $\mathbf{2b_1}$ dodecahedron. Their edges will cross each other at right angles at the midpoints.

Q4 The compound of icosahedron and dodecahedron in dual position shows that there is a dodecahedron vertex in the middle of each icosahedron face and an icosahedron vertex in the middle of each dodecahedron face. So the fact that one has 20 faces and 12 vertices while the other has 20 vertices and 12 faces is not a coincidence but a consequence of the fact that they are dual.

Q5 The dual has *m* vertices, *k* faces, and *n* edges. We can always imagine a compound with each polyhedron having one vertex inside each face of the other. There is a one-to-one mapping between the faces of one and the vertices of the other.

Q6 The dual also satisfies Euler's theorem. In the original, $V + F = E + 2$. In the dual, we would add $F + V = E + 2$, giving the same sums, just switching the order of the terms on the left side, that is, switching the number of faces with the number of vertices.

9.3 Dual Polyhedra

Challenge See the answer to Question 1.

Q1 The dual has ten kite-shaped faces. Each kite is made of two $\mathbf{b_1}$s and two $\mathbf{b_3}$s (or a scaled-up version of this). Corresponding to the top and bottom 5-sided face of the antiprism, the dual has a top and bottom vertex with five edges. Traditionally, this form is called a *trapezohedron* even though it is not made of trapezoids. (There is an older, British sense of the word *trapezoid* that means a quadrilateral with no parallel sides.)

Q2 The dual to an *n*-gon pyramid is another *n*-gon pyramid, but upside-down relative to the first one. In the self-dual pentagonal case, making the $\mathbf{2y_3}$ struts as $\mathbf{y_1}$-$\mathbf{y_3}$-$\mathbf{y_2}$ (or $\mathbf{y_1}$-$\mathbf{y_2}$-$\mathbf{y_3}$) puts a ball at the distance $\mathbf{y_1}$ from the base, which is where it crosses at right angles the edge of the dual pyramid.

The dual of a pentagonal pyramid is another pentagonal pyramid. It is self-dual.

2. Every *n*-gon pyramid is self-dual. The tetrahedron is the special case when $n = 3$. The stella octangula is the compound of a regular tetrahedron with its dual regular tetrahedron.

Q3 $v = \frac{1}{f}$ or $f = \frac{1}{v}$.

Q4 Generally, if two dual polyhedra are arranged with their edges tangent to a unit sphere, the distance from one's vertex to the center of the sphere is the reciprocal of the distance from the other's face to the center of the sphere. This relation is so handy that it is often convenient to normalize lengths so a polyhedron is *midscribed* to the unit sphere, meaning the edges are tangent to the sphere.

Explorations 9

A. Dual to (3, 3, 3, 3, 6) is a tessellation of irregular pentagons. Dual to (4, 8, 8) is a tessellation of isosceles right triangles.

B. The dual is the icosidodecahedron, which has 12 pentagons and 20 triangles. The compound can be made with $\mathbf{2b_2}$ edges for the icosidodecahedron and $\mathbf{r_1} + \mathbf{r_3}$ edges for the rhombic triacontahedron. The right-angle crossing occurs at the midpoint of the icosidodecahedron's edges, but it is not the middle of the rhombic triacontahedron's edges.

C. There is one with ten kites each consisting of two $\mathbf{r_1}$s (along the skew 10-gon) and two $\mathbf{y_3}$s (connecting to the pole). In another, the kites consist of two $\mathbf{y_1}$s and two $\mathbf{r_2}$s, in the corresponding places.

D. An *n*-gonal dipyramid—it is shaped like two *n*-gonal pyramids glued base-to-base. It has $2n$ isosceles triangles for faces and $n + 2$ vertices (*n* around the equator and 2 poles). The regular octahedron is a special case of such a 4-gonal dipyramid. In general, *n*-gonal dipyramids cannot be made with the Zome System.

E. The inside has an octahedron-shaped cavity, so a small spherical weight always lands inside one of the six vertices of the octahedron. It is positioned like the dual to the cube, with each octahedron vertex centered inside one of the six faces (marked with one to six dots). On the same principle, 20-faced, icosahedral spherical dice would have dodecahedron-shaped cavities.

10.1 Angular Deficit

Challenge The smallest sum is with three 36-degree angles, totaling 108. The largest sum is 360, formed by putting three struts anywhere in a common plane.

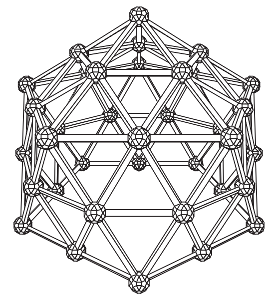
Q1 360 – 5(60) = 60

Q2 12(60) = 720

1. *See table below.*

Q3 The total angular deficit of any polyhedron is 720 degrees.

2.



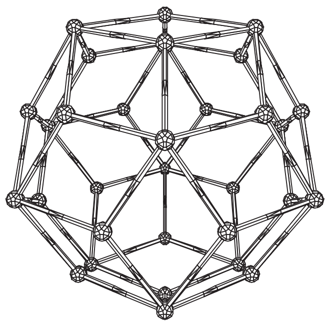
Zome partial icosahedral dome

Q4 At the vertices where six triangles meet, the angular deficit is zero. So only the five-fold vertices contribute to the sum of 12(60) = 720.

Q5 In any frequency icosahedron, there will be 12 five-fold vertices; the rest are six-fold, so the sum is 720.

Polyhedron	Number of vertices	Angular deficit	
		At each vertex	Total
cube	8	90	720
regular icosahedron	12	60	720
regular octahedron	6	120	720
regular dodecahedron	20	36	720
regular tetrahedron	4	180	720
triangular prism	6	120	720
pentagonal prism	10	72	720
n-gon prism	2*n*	$\frac{360}{n}$	720
pentagonal antiprism (with equilateral sides)	10	72	720
n-gon antiprism	2*n*	$\frac{360}{n}$	720

3.



Zome rhombic triacontahedron

Q6 **a.** $180 - x$

b. $12(360 - 5x) + 20(360 - 3(180 - x)) = 720$ (The x's drop out when you simplify the left side, so we cannot use this equation to solve for x. A method of determining x will be seen in Unit 13.)

Q7 **a.** $180 - 2x$

b. $5(360 - (108 + 2x)) + (360 - 5(180 - 2x)) = 720$. Again, the x's drop out.

Q8 $n(360 - (180 - \left(\frac{360}{n}\right) + 2x)) + (360 - n(180 - 2x)) = 720$. Here, n and x drop out.

10.2 Nonconvex Polyhedra

Q1 Both theorems hold for the indented polyhedron.

Q2 $V = 16, F = 16, E = 32$, so $V + F = E$. This is a variation of Euler's theorem that applies to all one-hole donuts.

Q3 At eight of the vertices, there are two 90-degree angles and two angles of $90 - x$ degrees, totaling to $360 - 2x$, which is a $2x$ deficit. At the other eight vertices, there are two 90-degree angles and two $90 + x$-degree angles, totaling to $360 + 2x$, which is a $-2x$ deficit. The total angular deficit is $8(2x) + 8(-2x) = 0$. The angular deficit on a one-hole donut polyhedron is always zero.

Q4 There are 15 vertices, 24 edges, and 12 faces; Euler's theorem is not satisfied. Descartes' theorem is not satisfied either since there are 14 vertices with an angular deficit of 90 and one with −180 for a total of 1080.

Q5 There are 16 vertices, 24 edges, and 12 faces; Euler's theorem is not satisfied. Descartes' theorem is not satisfied either since there are 16 vertices with an angular deficit of 90.

Explorations 10

A. Both theorems will apply to a polyhedron that is merely indented. If there are unusual angles, however, the student may have to wait until Unit 13 to find their exact measure.

B. Each vertex has a 108-degree angle from the pentagon and two 120-degree angles from the hexagons. They sum to $108 + 120 + 120 = 348$, leaving a deficit of $360 - 348 = 12$ degrees. The total deficit of 720 comes from adding up the 12s of the n vertices, so $n = \frac{720}{12} = 60$. One can verify this from the model because every pentagonal vertex is a different vertex of the polyhedron, so the 12 pentagons provide $12(5) = 60$ vertices.

C. It is not possible. Those three faces would contribute $60 + 90 + 108 = 258$ to each vertex, leaving an angular deficit of $360 - 258 = 102$. The number of vertices would have to be $\frac{720}{102}$, which is not an integer.

D. Since there are 120 identical vertices, each must have an angular deficit of $\frac{720}{120} = 6$ degrees. So, we need to find three regular polygons that have interior angles summing to 354 degrees. Searching among the Zome-constructible choices, we find a square, a hexagon, and a decagon give $90 + 120 + 144 = 354$. If you can't construct it, peek ahead to the truncated icosidodecahedron of Unit 12.

E. If the pentagons and hexagons are regular, there must be three edges per vertex since more faces would exceed 360 degrees. So, $V = \frac{5V_5 + 6V_6}{3}$. Substituting and solving for V_5, the V_6s cancel and we find $V_5 = 12$. In fact, the assumption of regular polygons is not essential here; if we assume only that every vertex has three faces, any polyhedron consisting of pentagons and hexagons must have exactly 12 pentagons.

11.1 One Cube

Q1 6 faces, 12 edges, 8 vertices for the cube; 12 faces, 30 edges, 20 vertices for the dodecahedron.

Q2 The angles are right angles because all of them are equal, as can be seen by inspecting the balls. The sides are equal, because they are all $\mathbf{b}_x$s.

Q3 The four-fold axes of the cube align with 3 of the dodecahedron's 15 two-fold axes.

11.2 Five Cubes

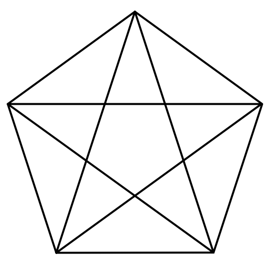
Challenge Yes, you can fit five cubes into the dodecahedron.

Q1 There is one square behind each of the dodecahedron's edges. Since it has 30 edges, there are 30 squares.

Q2 The 30 squares, at 6 squares per cube, make 5 concentric cubes.

Q3 Six of the second cube's edges would intersect some edge of the first cube.

Q4



Pentagram inside a pentagon

Q5 The edges are of length
$\mathbf{b}_2 + \mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b}_2 + \mathbf{b}_3 = \mathbf{b}_4$.
(See Unit 7 on the golden ratio.)

Q6 Each cube has 4 three-fold axes of symmetry. Each of the four aligns with 1 three-fold axis of one of the other four cubes, where the two cubes share a vertex. Five cubes times 4 three-fold axes per cube would give 20 axes, but, because they superimpose in pairs, divide by 2 to find there are only ten directions, which account for all ten of the dodecahedron's three-fold axes. The 3 four-fold axes of each cube align with 3 of the dodecahedron's two-fold axes. Each cube relates to a different set of three, accounting for all 15 of the dodecahedron's two-fold axes. The five-fold axes of the dodecahedron do not align with any aspect of the cube's symmetry, and the two-fold axes of the cubes do not align with any aspect of the dodecahedron's symmetry.

11.3 Related Constructions

Challenge The vertices can be chosen so that the struts of the five tetrahedra do not intersect each other, so no scaling up is necessary.

1. A $\mathbf{g}_1$ tetrahedron is built from four of the vertices of a $\mathbf{b}_1$ cube. This was the scaffolding method of constructing the regular tetrahedron in Unit 3.

2. A $\mathbf{g}_2$ tetrahedron is built in a $\mathbf{b}_1$ dodecahedron. Depending on which tetrahedron in the cube in the dodecahedron is chosen, there are two mirror-image ways of doing this.

Q1 There are two ways; one is the mirror image of the other.

4. The $2\mathbf{b}_1$ icosahedron has its 12 vertices lying on the 12 edges of the $\mathbf{g}_1 + \mathbf{g}_2$ octahedron.

Q2 An icosahedron in a tetrahedron, because the tetrahedron is self-dual and now appears on the outside.

5. Simply extend the octahedron into a $3\mathbf{g}_3$ tetrahedron by adding tetrahedra to four of its faces. There are two mirror-image ways to do this, depending on which four faces you pick.

Explorations 11

A. If the roof doesn't fit, try turning it 90 degrees.

B. This makes clear that the faces of the rhombic triacontahedron appear in five groups of six. Any group of six is like the planes of the cube—opposite faces parallel and other pairs perpendicular.

E. Yes, a $\mathbf{g}_2$ octahedron can be inscribed in a $2\mathbf{b}_1$ icosahedron. The octahedron vertices are six of the icosahedron's edge midpoints.

F. The model has eight spikes, pointing to the corners of a cube. The inner 12 vertices are positioned as in an icosahedron.

G. The green 60-degree angles are the tetrahedron points; the blue ones are the concave 5-way indents. The two-fold axes pass through the $\mathbf{g}_1$'s midpoints.

12.1 The Icosidodecahedron

Challenge The pentagonal pyramid and the pentagonal antiprism can be constructed with only regular pentagons and equilateral triangles. The pentagonal antiprism can be obtained by removing two opposite pyramids from an icosahedron. This gives a polyhedron with ten triangles and two pentagons. You can instead remove two pyramids that are not opposite to make a different polyhedron with ten triangles and two pentagons. (This is analogous to an *isomer* in chemistry—a different arrangement of the same components.) Another possibility is to remove only one pyramid. Yet another is to remove three pyramids, giving a polyhedron with three pentagons and five triangles. These are all varieties of *diminished icosahedra*. The remaining Zome-constructible convex polyhedron consisting of pentagons and triangles is the icosidodecahedron. Others could be built with paper; for example, cut an icosidodecahedron in half along a decagon cross section and glue the halves together after rotating one 36 degrees.

Q1 There are 20 triangles and 12 pentagons.

Q2 The Greek prefixes for 20 (icosi-) and 12 (dodeca-) are combined. Also, the faces are in planes corresponding to those two Platonic solids, and they appear in the same numbers and in the same relative positions.

Q3 There are 60 edges. Here are some counting strategies:

1. Each edge is part of a unique pentagon, so 12 pentagons times 5 edges each gives 60 edges.

2. Each edge is part of a unique triangle, so 20 triangles times 3 edges each gives 60 edges.

3. See it as six large regular 10-gons (six equators); 6 times 10 gives 60 edges.

Q4 There are many ways to count the 30 vertices. One involves combinations: Any two of the six decagon equators cross twice (at an opposite pair of vertices), so there are ${}_6C_2$ pairs of vertices—that is, $\frac{6!}{(6-2)!\,2!}$ pairs—and 15 times 2 equals 30 vertices.

Q5 Five-fold axes pass through the centers of opposite pentagons, and three-fold axes pass through the centers of opposite triangles. The two-fold axes pass through opposite vertices. There are 15 mirror planes.

Q6 It is the same symmetry, called icosahedral symmetry.

12.2 Archimedean Solids and Notation

Challenge Prisms, antiprisms, and the Archimedean solids listed in Questions 2, 3 and 5.

Q1 **a.** The vertex at the top of the pyramid is not equivalent to the other vertices.

b. They contain only one type of polygon.

c. Rectangles are not regular 4-gons.

Q2 The *n*-gon prism is (4, 4, *n*).

Q3 The *n*-gon antiprism is (3, 3, 3, *n*).

Q4 (6, 6, 6): 360 degrees of interior angles at a vertex makes a tessellation.
(8, 8, 8): More than 360 degrees of angle at a vertex is impossible.

Q5 The 13 Archimedean solids are: truncated tetrahedron (3, 6, 6); truncated octahedron (4, 6, 6); truncated cube (3, 8, 8); truncated dodecahedron (3, 10, 10); truncated icosahedron (5, 6, 6); cuboctahedron (3, 4, 3, 4); icosidodecahedron (3, 5, 3, 5); rhombicuboctahedron (sometimes called the *small rhombicuboctahedron*) (3, 4, 4, 4); truncated cuboctahedron (sometimes called the *great rhombicuboctahedron*) (4, 6, 8); snub cube (3, 3, 3, 3, 4); rhombicosidodecahedron (sometimes called the *small rhombicosidodecahedron*) (3, 4, 5, 4); truncated icosidodecahedron (sometimes called the *great rhombicosidodecahedron*) (4, 6, 10); and snub dodecahedron (3, 3, 3, 3, 5).

12.3 Archimedean Solids in the Zome System

Challenge The image by Kepler shows the final forms.

Q1 *See table on page 231.*

12.3, Q1 *(from page 230)*

	Polyhedron	Faces	Edges	Vertices	5-fold axes	4-fold axes	3-fold axes	2-fold axes	Mirrors	Symmetry of the
a.	*★truncated tetrahedron,* (3, 6, 6)	8	18	12	0	0	4	3	6	tetrahedron
b.	*★truncated cube,* (3, 8, 8)	14	36	24	0	3	4	6	9	octahedron
c.	*★truncated octahedron,* (4, 6, 6)	14	36	24	0	3	4	6	9	octahedron
d.	*truncated icosahedron,* (5, 6, 6)	32	90	60	6	0	10	15	15	icosahedron
e.	*truncated dodecahedron,* (3, 10, 10)	32	90	60	6	0	10	15	15	icosahedron
f.	*★cuboctahedron,* (3, 4, 3, 4)	14	24	12	0	3	4	6	9	octahedron
g.	*icosidodecahedron,* (3, 5, 3, 5)	32	60	30	6	0	10	15	15	icosahedron
h.	*★rhombicuboctahedron,* (3, 4, 4, 4)	26	48	24	0	3	4	6	9	octahedron
i.	*rhombicosidodecahedron,* (3, 4, 5, 4)	62	120	60	6	0	10	15	15	icosahedron
j.	*★truncated cuboctahedron,* (4, 6, 8)	26	72	48	0	3	4	6	9	octahedron
k.	*truncated icosidodecahedron,* (4, 6, 10)	62	180	120	6	0	10	15	15	icosahedron
l.	*snub cube,* (3, 3, 3, 3, 4)	38	60	24	0	3	4	6	0	rotations of octahedron
m.	*snub dodecahedron,* (3, 3, 3, 3, 5)	92	150	60	6	0	10	15	0	rotations of icosahedron

Explorations 12

A. Three tessellations are like the Platonic solids, but planar: {4, 4}, {6, 3}, and {3, 6}. In the two-dimensional version of truncation, truncating {4, 4} gives (4, 8, 8); truncating {6, 3} gives (3, 12, 12); but truncating {3, 6} gives {6, 3} again. Truncating {6, 3} or {3, 6} to the edge midpoints gives (3, 6, 3, 6); but truncating {4, 4} to the edge midpoints gives {4, 4} again (rotated 45 degrees).

B. From the icosidodecahedron, you get a compound with the rhombic triacontahedron (30 red rhombi). From the cuboctahedron, you get a compound with the rhombic dodecahedron (12 yellow rhombi). From a $\mathbf{2g_1}$ cuboctahedron, the resulting rhombic dodecahedron has length $\mathbf{3y_1}$, with edges divided at the one-third point.

C. The faces in the plane of a tetrahedron will be ones with three-fold symmetry—for example, four triangles of the truncated cube or four hexagons of the truncated icosahedron. You can build a large regular tetrahedron, place any Archimedean solid inside it, with a triangle or hexagon down, and rotate it until three other faces (triangles or hexagons) are parallel to the top three faces of the tetrahedron.

D. Expanding the cube or octahedron gives the rhombicuboctahedron. Expanding the icosahedron or dodecahedron gives the rhombicosidodecahedron. Expanding the tetrahedron gives the cuboctahedron. The icosidodecahedron can be expanded using red struts to make 60 rectangles, each surrounded by a triangle, a rhombus, a pentagon, and another rhombus.

E. The rhombicuboctahedron has regular 8-gon slices, and the rhombicosidodecahedron has regular 10-gon slices. In the first case an isomer is possible by twisting one cap 45 degrees, and in the second case there are several isomers as one or two caps can be twisted.

F. The model consists of twelve 5-gons, twenty 3-gons, and sixty isosceles right triangles.

G. The cuboctahedron and icosidodecahedron are quasi-regular. The first is the intersection of the cube and octahedron when their dual pair is constructed. The second is the intersection of the icosahedron and dodecahedron when their dual pair is constructed.

H. The *snub tetrahedron* is (3, 3, 3, 3, 3), which is {3, 5}, the icosahedron.

13.1 Lengths

Challenge $\mathbf{r_1} = \frac{\sqrt{2+\tau}}{2}$ units; $\mathbf{y_1} = \frac{\sqrt{3}}{2}$ units

Q1 **a.** $\tau\mathbf{b_1}$

b. $\tau^2\mathbf{b_1}$

Q2 **a.** golden

b. $\frac{1+\sqrt{5}}{2}$

c. approximately 1.61803

Q3 From the isosceles right triangles you know $\mathbf{g_1} = \sqrt{2}\mathbf{b_1}$ or $\sqrt{2}$ units, and $\mathbf{g_2} = \sqrt{2}\,\mathbf{b_2} = \tau\sqrt{2}\mathbf{b_1}$, so $\mathbf{g_2} = \tau\sqrt{2}$ units.

1. The long diagonal of a $\mathbf{b_1}$ cube can be constructed as $\mathbf{2y_1}$.

Q4 A face diagonal has length $\mathbf{g_1} = \sqrt{2}\,\mathbf{b_1}$. The cube's diagonal has length $\sqrt{(\mathbf{b_1})^2 + (\sqrt{2}\mathbf{b_1})^2} = \sqrt{3}\mathbf{b_1}$, so

$$\mathbf{y_1} = \left(\frac{\sqrt{3}}{2}\right)\mathbf{b_1}$$

Q5 $\mathbf{y_2} = \tau\mathbf{y_1} = \left(\frac{\tau\sqrt{3}}{2}\right)\mathbf{b_1}$, and $\mathbf{y_3} = \tau^2\mathbf{y_1} = \left(\frac{\tau^2\sqrt{3}}{2}\right)\mathbf{b_1}$.

Q6 A right triangle with legs $\mathbf{b_1}$, and $\mathbf{b_3}$ has hypotenuse $\mathbf{2y_2}$. So $(2\mathbf{y_2})^2 = \mathbf{b_1}^2 + (\mathbf{b_1}\tau^2)^2$. Solving for $\mathbf{y_2}$ gives

$$\mathbf{y_2} = \left(\frac{\sqrt{1+\tau^4}}{2}\right)\mathbf{b_1}$$

Q7 It follows from $\mathbf{b_3} = \tau^2\mathbf{b_1}$, $\mathbf{b_2} = \tau\mathbf{b_1}$, and $\mathbf{b_3} = \mathbf{b_2} + \mathbf{b_1}$.

Q8 *See table on page 233.*

Q9 You see a Fibonacci sequence in the second and third columns. The columns are shifted by one row relative to each other. Each entry is the sum of the two entries above it.

Q10 From the table, $\tau^4 = 2 + 3\tau$. So you can simplify:

$$\mathbf{y_2} = \frac{\sqrt{1+\tau^4}}{2}\mathbf{b_1} = \frac{\sqrt{3+3\tau}}{2}\mathbf{b_1} = \frac{\sqrt{3(1+\tau)}}{2}\mathbf{b_1}$$

$$= \frac{\sqrt{3(\tau^2)}}{2}\mathbf{b_1} = \frac{\tau\sqrt{3}}{2}\mathbf{b_1}$$

13.1, Q8 *(from page 232)*

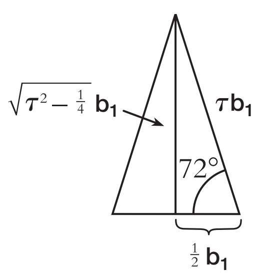
Power of τ	Constant	Coefficient of τ	Expression
τ^{-2}	2	–1	$2 - \tau$
τ^{-1}	–1	1	$-1 + \tau$
τ^{0}	1	0	1
τ^{1}	0	1	τ
τ^{2}	1	1	$1 + \tau$
τ^{3}	1	2	$1 + 2\tau$
τ^{4}	2	3	$2 + 3\tau$
τ^{5}	3	5	$3 + 5\tau$
τ^{6}	5	8	$5 + 8\tau$

3. A right triangle with legs $\mathbf{b_1}$ and $\mathbf{b_2}$ has hypotenuse $\mathbf{2r_1}$.

Q11 So $(2\mathbf{r_1})^2 = \mathbf{b_1}^2 + (\mathbf{b_1}\tau)^2$. Solving for $\mathbf{r_1}$ gives

$$\mathbf{r_1} = \frac{\sqrt{1+\tau^2}}{2}\mathbf{b_1} = \frac{\sqrt{2+\tau}}{2}\mathbf{b_1}$$

Q12



Right triangles with hypotenuse $\tau\mathbf{b_1}$

Since $\mathbf{b_1}$ and $\mathbf{b_2}$ are in the red plane, the angle between them is 72 degrees;

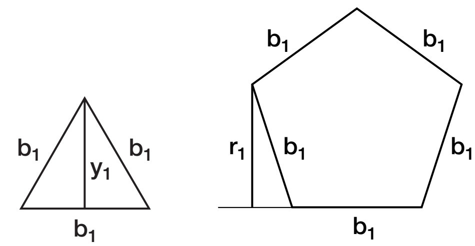
$$\sin 72 = \frac{\sqrt{\tau^2 - \left(\frac{1}{2}\right)^2}}{\tau}$$

Q13 $$\sin 72 = \frac{\sqrt{\tau^2 - \left(\frac{1}{2}\right)^2}}{\tau} = \frac{\sqrt{4\tau^2 - 1}}{2\tau} = \frac{\sqrt{4 - \tau^{-2}}}{2}$$
$$= \frac{\sqrt{4 - (2 - \tau)}}{2} = \frac{\sqrt{2 + \tau}}{2}$$

So $(\sin 72)\,\mathbf{b_1}$ agrees with the value for $\mathbf{r_1}$ found in Question 11.

Q14 $\mathbf{r_1} = (\sin 72)\,\mathbf{b_1}$, and $\mathbf{y_1} = (\sin 60)\,\mathbf{b_1}$.

5.



Triangle and pentagon
with dropped perpendiculars

Q15 Yes, the exterior angle of the pentagon is 72 degrees; $\sin 72 = \frac{\mathbf{r_1}}{\mathbf{b_1}}$.

Q16 A Zome model shows that the ratio of diagonal to edge in a dodecahedron is $\frac{2\mathbf{y_2}}{\mathbf{b_1}}$. Knowing

$$\mathbf{y_2} = \tau\mathbf{y_1} = \frac{\tau\sqrt{3}}{2}\mathbf{b_1} = \frac{(1+\sqrt{5})\sqrt{3}}{4}\mathbf{b_1}$$

you get

$$\frac{2\mathbf{y_2}}{\mathbf{b_1}} = \frac{\sqrt{3} + \sqrt{15}}{2}$$

as the diagonal-to-edge ratio. For a 10-meter edge, there is a $5(\sqrt{3} + \sqrt{15})$ meter diagonal. For engineering or construction purposes, an approximate numerical answer is usually desired, which is determined with a calculator as 28 meters.

13.2 Angles

Challenge See Questions 1–8.

Q1 The angles are 36°, 72°, 60°, 90°, 108°, 120°, 144°, and 180°.

Q2 The angle between a red pole and any of the ten blue struts in the red plane is a right angle. The angle between a yellow pole and any of the six blue struts in the yellow plane is also a right angle. These can also be seen as a right angle between a blue pole and a red or yellow strut in the blue plane.

2. A right triangle with legs $\mathbf{b_1}$ and $\mathbf{b_3}$ has hypotenuse $\mathbf{2y_2}$, and a right triangle with legs $\mathbf{b_1}$ and $\mathbf{b_2}$ has hypotenuse $\mathbf{2r_1}$.

Q3 The most convenient way to characterize α and β is by their tangent. From the triangles, you see α has tangent $\frac{1}{\tau^2}$ and β has tangent $\frac{1}{\tau}$. Writing $\arctan(x)$

for the arctangent function (the angle with tangent x), you have $\alpha = \arctan\frac{1}{\tau^2}$ and $\beta = \arctan\frac{1}{\tau}$. Hence, $\alpha \approx 20.91$ degrees and $\beta \approx 31.72$ degrees. (We use fractions of a degree rather than minutes and seconds.)

Q4 The interior angles of a red rhombus are 2β and $180 - 2\beta$, approximately 63.44 and 116.56 degrees.

Q5 The interior angles of that yellow rhombus are 2α and $180 - 2\alpha$, approximately 41.81 and 138.19 degrees.

Q6 The angle between an adjacent red and yellow strut is $90 - \alpha - \beta$, approximately 37.38 degrees.

Q7 Let an altitude bisect the yellow-yellow angle. You have a right triangle with hypotenuse $\frac{\sqrt{3}}{2}$ and one leg $\frac{1}{2}$. By the Pythagorean theorem, the other leg must be $\frac{\sqrt{2}}{2}$, and $\gamma = \arctan\sqrt{2}$, approximately 54.74 degrees.

Q8 It can be described as $2(90 - \gamma)$ or $2 \arctan\frac{\sqrt{2}}{2}$, approximately 70.53 degrees.

Q9 γ

Q10 2α

Q11 2β

Q12 $180 - 2\gamma$

Q13 2γ

4. Cut the rhombus in half one way to get a $2b_1$–$2y_1$–$2y_1$ isosceles triangle, and cut the other way to get a $2g_1$–$2y_1$–$2y_1$ isosceles triangle. One quarter of the rhombus is a b_1–g_1–$2y_1$ right triangle. Two right triangles assemble to form a b_1-by-g_1 rectangle, where the ratio of the sides is $\sqrt{2}$.

5. Substituting a gb_1 for the g_1 gives a square in this plane, which appears in some of the Archimedean solids.

6. The green plane is the plane of any rhombus from the rhombic dodecahedron.

13.3 Dihedral Angles

Challenge

Polyhedra	Dihedral angles
tetrahedron	≈ 70.5°
octahedron	≈ 109.5°
dodecahedron	≈ 116.6°
icosahedron	≈ 138.2°

Q1 60 degrees

Q2 Half the above; 30 degrees

Q3 Either 30 or 150 degrees, depending on whether one takes the larger or smaller angle. The two possibilities will always be supplementary.

Q4 The angle between the normals is supplementary to the dihedral angle. This is because the angle between the normals, plus the angle between the planes, plus 2 times 90 degrees, add up to 360 degrees, as you can see by sighting along the hinge.

Q5 The dihedral angle of the tetrahedron is $180 - 2\gamma$, approximately 70.5 degrees.

6. The icosahedron's face normals are yellow lines, three-fold symmetry axes.

Q6 The face normals are separated by 2α, so the dihedral angle is $180 - 2\alpha$, approximately 138.2 degrees.

Q7 The dodecahedron's face normals are red lines, five-fold symmetry axes. The face normals are separated by 2β, so the dihedral angle is $180 - 2\beta$, approximately 116.6 degrees.

Q8 The regular octahedron and the regular tetrahedron have dihedral angles that are supplementary. The tetrahedron's is $180 - 2\gamma$, so the octahedron's is 2γ.

Q9 The dihedral angles are 60, 72, 108, and 120 degrees. To construct the normals, rest the parallelepiped with a face flat on the table; you can insert a blue strut vertically into any of the top four balls. The blue strut is a normal for the top and bottom face. Do this for each of two faces.

Explorations 13

A. There is enough flexibility in the components to make a triangular structure with those lengths. The zomeballs have holes in approximately the directions needed. However, these lengths are not the lengths of the sides of a right triangle, because they do not satisfy the Pythagorean theorem. A calculator shows that $(\mathbf{b_3})^2 + (\mathbf{4b_1})^2$ is approximately $22.85(\mathbf{b_1})^2$ while $(\mathbf{5r_1})^2$ is approximately $22.61(\mathbf{b_1})^2$. So $A^2 + B^2 \neq C^2$, and the struts must be slightly bent.

B. $\tau^k = F_{k-1} + F_k\tau$. Proof: The base case of $k = 1$ is evident. For the general case, we make the induction hypothesis that $\tau^k = F_{k-1} + F_k\tau$, and we need to prove $\tau^{k+1} = F_k + F_{k+1}\tau$. Using the hypothesis, we write $\tau^{k+1} = \tau\tau^k = \tau(F_{k-1} + F_k\tau)$. Using the $\tau^2 = 1 + \tau$ property, this can be expanded as $F_{k-1}\tau + F_k\tau^2 = F_{k-1}\tau + F_k(1 + \tau) = F_k + (F_{k-1} + F_k)\tau = F_k + F_{k+1}\tau$. The last step uses the definition of the Fibonacci sequence.

C. You know $\beta = \arctan\frac{1}{\tau}$, or equivalently, $\tan\beta = \frac{1}{\tau}$. Using the formula for $\tan 2x$,

$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta}$$

$$= \frac{2\frac{1}{\tau}}{1 - \left(\frac{1}{\tau}\right)^2}$$

$$= \frac{2\tau}{\tau^2 - 1} = 2$$

So $2\beta = \arctan 2$, and $\beta = \frac{\arctan 2}{2}$.

D. Every blue angle shows up as an angle between some pair of edges of both the dodecahedron and the icosahedron. The edges in both cases correspond to all the two-fold axes, every blue line.

E. For the $\mathbf{b_1}$–$\mathbf{g_2}$–$\mathbf{g_2}$ triangle, the two $\mathbf{g_2}$s want to be in the same red hole, but you can verify the construction by replacing one with the $\mathbf{b_2}$ legs of the right triangle of which it is the hypotenuse. (The blue legs are not in the plane of the original triangle.) The apex angles are

$$2\arcsin\left(\frac{\tau^i}{2\sqrt{2}}\right)$$

where i is −1, 0, or 1, approximately 25.24, 41.41, and 69.79 degrees. These three triangles are each in a different plane, but they are not planes perpendicular to any of the struts.

F. In the $\mathbf{g_1}$–$\mathbf{g_2}$–$\mathbf{2b_1}$ triangle that sits in the corners of each octahedron face of the model, let $A = 2$, $B = \tau\sqrt{2}$, and $C = \sqrt{2}$. Solve for $\cos\theta$ in the law of cosines, and use a calculator to find that θ is approximately 37.76 degrees. The same method can be used to find any other green angles of interest, after first building some triangle that involves the angle.

14.1 Rhombic Zonohedra

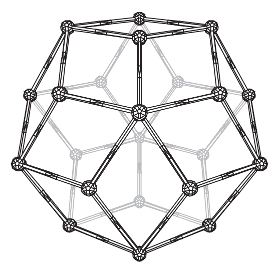
Challenge Most of the polyhedra in this unit are composed only of parallelograms.

1. The acute one has two vertices where three acute angles meet. The obtuse one has two vertices where three obtuse angles meet.

3. There are several choices. With yellow, you can combine the two shapes of rhombus in several ways. In blue, you can combine two or three shapes of rhombus. Notice that opposite faces are always equal, so six faces can include at most three rhombic shapes.

Q1 A rhombic hexahedron that is not a rhombohedron must have two shapes of rhombus.

5.



Five-fold polar zonohedron

Q2 20 faces, with two different shapes (except in the case of the red version, where there is only one shape)

14.2 Zones

Challenge All trips have the same length.

Q1 Answers will vary. See table in Question 4.

Q2 Because there are only two pairs of opposite parallel sides

Q3 $2(n - 1)$, since each other zone is crossed twice

Q4

Zonohedra	Number of zones (edge directions)	Number of faces
rhombohedra, parallelepipeds	3	6
rhombic dodecahedra	4	12
five-fold polar zonohedra	5	20
rhombic triacontahedron	6	30

Q5 These *n*-zone zonohedra have $n(n-1)$ parallelogram faces.

Q6 There are *n* zones, each with $2(n-1)$ faces, so you have a total of $2n(n-1)$ faces. However, each face has been counted twice, so there are $n(n-1)$ faces altogether.

Q7 Six of the twelve faces would be removed and the caps would join into a rhombohedron.

Q8 The red five-fold polar zonohedron

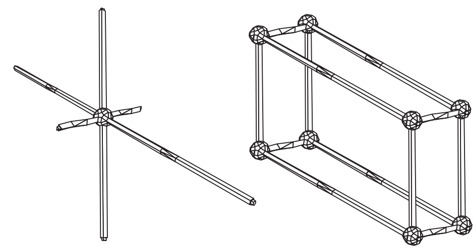
Q9 The rhombic dodecahedron of the second kind

Q10 The opposite sides of each face are parallel, so zones can be stretched to produce a wide variety of forms, for example, in architectural applications.

14.3 Stars

1. For the (yellow) rhombic dodecahedron, the eight struts are arranged as if they were pointing toward the vertices of a cube. For the (red) rhombic dodecahedron of the second kind, the eight red struts fill all the pentagonal holes of the zomeball except for two opposite pairs of holes; they point toward eight of the vertices of an icosahedron.

2. The parallelepiped will depend on the star chosen. The figure shows one possibility.



Zome parallelepiped and corresponding star

3. Answers will vary.

Q1 The rhombic triacontahedron

Q2 There are 20 yellow holes, making ten directions. With ten zones, $2(n-1)$ gives us 18 faces per zone, and $n(n-1)$ gives us 90 faces. All the faces are quadrilaterals, so there are 4 times 90 divided by 2, or 180 edges.

Q3–Q4, 6. *See table on page 237.*

Explorations 14

A. **a.** Cube, regular octahedron

b. Regular icosahedron, regular dodecahedron

c. With all the diagonals, the result is a compound of two dual Platonic polyhedra.

B. If the acute angle of the rhombus is less than 60 degrees, the obtuse angle is greater than 120 degrees. For such a rhombus, one cannot place three obtuse angles together at a vertex to make the obtuse rhombohedron, as the sum would exceed 360 degrees. Only the acute (pointy) rhombohedron exists for rhombi with an angle less than 60 degrees. Two rhombohedra exist for any rhombus with an acute angle between 60 and 90 degrees.

C. There are four blue rhombus shapes possible, with acute angles of 36, 60, 72, or 90 degrees. The 90-degree rhombus is a square, which leads to the cube as one rhombohedron. With the 36-degree angle, the only possibility is an acute rhombohedron (because it is less than 60 degrees), and it is easily constructed. The 60-degree rhombus also does not allow an obtuse rhombohedron (because three 120-degree angles meeting would total 360 degrees exactly). In theory there is an acute rhombohedron with 60-degree rhombi, and it is easy to make from

14.3, Q3–Q4, 6. *(from page 236)*

Q3	Q4	6
Truncated octahedron	6 zones	The star consists of the 6 two-fold axes of the cube.
Truncated cuboctahedron	9 zones	The star consists of the 6 two-fold and 3 four-fold axes of the cube.
Truncated icosidodecahedron	15 zones	The star is the blue starburst, which is the 15 two-fold axes of the icosahedron.

paper. However, when you try to make it with the Zome System, you see that the zomeballs do not allow placing three blue 60-degree angles together, so it cannot be built. With the 72-degree rhombi, it is easy to build the obtuse rhombohedron. However, the acute 72-degree rhombohedron, like the 60-degree one, though constructible in paper, is not Zome-constructible. In total, there are only three blue Zome rhombohedra.

D. The process still works, making a polyhedron that is flattened onto a plane, but struts have to be bent slightly to cross each other. This is the projection of a polar zonohedron onto the plane, except that the two zomeball poles push against each other, each wanting to be at the center.

E. There are five green three-fold polar zonohedra. In the hole closer to the pole, there are three different ways to insert the green strut. (The other two are mirror images that give the same angles.) In the hole closer to the equator, there are two ways. (Two others lead to "flat zonohedra," and one is a mirror image.)

F. These do not have exact six-fold or ten-fold symmetry, but are very close. With $\mathbf{r_1}$s and $\mathbf{y_1}$s, the six-fold form is about the size and shape of a football. It stands up nicely on a base of three $\mathbf{b_1}$s. The ten-fold forms both have groups of coplanar ribs, so groups of three faces are coplanar.

G. To count the number of regular skew *n*-gons, you need to count only the number of ways of making symmetric "umbrella ribs," as in Exploration E.

H. The rhombohedra have 1 three-fold axis, 3 two-fold axes, and 3 mirror planes arranged as in the triangular antiprism. (The special case of the cube also has additional symmetry, of course.) The rhombic dodecahedron has the same symmetry as the cube. The rhombic dodecahedron of the second type has the same symmetry as the *golden brick:* 3 mirrors and 3 two-fold axes. The five-fold polar zonohedra have the same symmetry as a pentagonal antiprism. The rhombic triacontahedron and enneacontahedron have the same symmetry as the icosahedron.

I. You can generate nonconvex zonohedra.

J. The figure is displayed at the end of the Explorations. With green struts you can make a 61-zone zonohedron.

15.1 Calculating Area

Challenge A parallelogram with sides $\mathbf{b_1}$ and $\mathbf{y_1}$, and angle α, has area $\mathbf{b_1}(\mathbf{y_1})(\sin \alpha)$, or approximately $0.309\mathbf{b_1}^2$. Or a $\mathbf{b_2}$–$\mathbf{y_1}$–$\mathbf{y_1}$ triangle with area $0.25\mathbf{b_1}^2$.

Q1 Recognize that this is an isosceles triangle with angles of 36, 36, and 108. The formula half the base times height provides its area if you first choose one side to be the base and then work out the height. Because of the bilateral symmetry, it is convenient to use $\mathbf{b_2}$ as the base, as the altitude bisects the obtuse angle. The height is then (sin 36) $\mathbf{b_1}$, but what is that explicitly? Apply the Pythagorean theorem to find the height is

$$\sqrt{1 - \frac{\tau^2}{4}}\mathbf{b_1}$$

Using the relation $\tau^2 = 1 + \tau$, the area can be simplified to

$$\tau\frac{\sqrt{3-\tau}}{4}\mathbf{b_1}^2$$

Q2 $s = \left(1 + \frac{\tau}{2}\right)\mathbf{b_1}$

Area $= \sqrt{\left(1+\frac{\tau}{2}\right)\mathbf{b_1}\left(\frac{\tau}{2}\right)\mathbf{b_1}\left(\frac{\tau}{2}\right)\mathbf{b_1}\left(1-\frac{\tau}{2}\right)\mathbf{b_1}} = \tau\frac{\sqrt{3-\tau}}{4}\mathbf{b_1}^2$ as above.

Q3 This isosceles triangle has angles 72, 72, and 36. Choosing the short side as base, the height is $\sqrt{\tau^2 - \frac{1}{4}}\mathbf{b_1}$. The semiperimeter is $s = \left(\frac{1}{2} + \tau\right)\mathbf{b_1}$. Both methods give the area as

$$\frac{\sqrt{\frac{3}{4}+\tau}}{2}\mathbf{b_1}^2$$

Q4 Inserting two diagonals, the $\mathbf{b_1}$ pentagon is seen to be composed of two triangles of the first type above and one of the second type. So its area is

$$\left(\frac{\tau\sqrt{3-\tau}}{4} + \frac{\sqrt{\frac{3}{4}+\tau}}{2}\right)\mathbf{b_1}^2$$

With some difficult manipulation, this can be simpified to

$$\frac{\sqrt{25+10\sqrt{5}}}{4}\mathbf{b_1}^2$$

(There is no need to do so, but you can verify this with a calculator if you wish.)

Q5 Inserting ten spokes from its center, the $\mathbf{b_1}$ decagon is seen to be composed of ten of the 36-72-72 triangles. So its area is

$$5\sqrt{\frac{3}{4}+\tau}\,\mathbf{b_1}^2$$

Q6 An $\mathbf{r_1}$ rhombus has diagonals $\mathbf{b_1}$ and $\mathbf{b_2}$, so it can be dissected into four right triangles, each with legs $\frac{1}{2}\mathbf{b_1}$ and $\frac{1}{2}\mathbf{b_2}$. Area $= 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\mathbf{b_1}\frac{\tau}{2}\mathbf{b_1} = \frac{\tau}{2}\mathbf{b_1}^2$.

Q7 The area is one half the product of the diagonals. A rhombus with diagonals x and y can be dissected into four right triangles, each with legs $\frac{x}{2}$ and $\frac{y}{2}$. Area $= 4\frac{1}{2}\frac{x}{2}\frac{y}{2} = \frac{xy}{2}$.

Q8 The skinny $\mathbf{y_1}$ rhombus has diagonals $\mathbf{b_0}$ and $\mathbf{b_2}$, so its area is $\frac{1}{2}\frac{1}{\tau}(\mathbf{b_1})(\tau)(\mathbf{b_1}) = \frac{1}{2}\mathbf{b_1}^2$. (Although you have no $\mathbf{b_0}$ struts, you can make a $\mathbf{y_2}$ rhombus, see it has a $\mathbf{b_1}$ diagonal, and scale down.) The fatter $\mathbf{y_1}$ rhombus has diagonals $\mathbf{b_1}$ and $\sqrt{2}\mathbf{b_1}$, so its area is $\frac{\sqrt{2}}{2}\mathbf{b_1}^2$. The $\sqrt{2}$ diagonal (green strut) is determined by applying the Pythagorean theorem, remembering that $\mathbf{y_1} = \frac{\sqrt{3}}{2}\mathbf{b_1}$.

15.2 Scaling Area

Challenge There are two possible solutions: $\mathbf{b_1}$ by $\mathbf{y_3}$, $\mathbf{b_2}$ by $\mathbf{y_2}$, and $\mathbf{b_3}$ by $\mathbf{y_1}$, or $\mathbf{b_1}$ by $\mathbf{r_3}$, $\mathbf{b_2}$ by $\mathbf{r_2}$, and $\mathbf{b_3}$ by $\mathbf{r_1}$. The key property in both cases is that as one side is scaled up by τ, the other side is scaled down by τ, leaving the product of the two sides unchanged. In the first case, the area is $\tau^2\mathbf{b_1}\mathbf{y_1}$; and in the second case, it is $\tau^2\mathbf{b_1}\mathbf{r_1}$. No red-yellow solution exists because there is no red-yellow right angle.

Q1 **a.** No

b. It gets multiplied by 3.

Q2 **a.** No

b. It gets multiplied by 3.

Q3 **a.** Yes

b. It gets multiplied by 9.

Q4 The ratio of areas equals the square of the scaling factor.

Q5 It is τ^2 times as big.

Q6 9

Q7 $A = \frac{1}{2}bh$

$A' = \frac{1}{2}kb\,kh = k^2(\frac{1}{2}bh) = k^2A$

There is an analogous method using Heron's formula.

Q8 This is the 36-36-108 triangle with lengths scaled up by τ, so you can take our answer from the previous activity and scale the area up by τ^2. The area is

$$\tau^3\frac{\sqrt{3-\tau}}{4}\mathbf{b_1}^2$$

Q9 First consider a $\mathbf{b_1}$ icosahedron. Sum 20 $\mathbf{b_1}$ equilateral triangles. Using Heron's formula (with $s = \frac{3}{2}\mathbf{b_1}$), each triangle has area

$$\sqrt{\frac{3}{2}\mathbf{b_1}\frac{1}{2}\mathbf{b_1}\frac{1}{2}\mathbf{b_1}\frac{1}{2}\mathbf{b_1}} = \frac{\sqrt{3}}{4}\mathbf{b_1}^2$$

So the $\mathbf{b_1}$ icosahedron has total surface area $5\sqrt{3}\mathbf{b_1}^2$. Our 3-meter icosahedron is scaled up by a linear factor of $3\frac{\text{meters}}{\mathbf{b_1}}$, so its area is scaled up by a factor of $9\frac{\text{meter}^2}{\mathbf{b_1}^2}$, making its area $45\sqrt{3}\,\text{meter}^2$. If you were thinking of how much paint to buy, you would probably want an approximate numerical answer, which a calculator shows is 77.9 meter^2.

Q10 First consider a $\mathbf{b_1}$ dodecahedron. Composed of 12 $\mathbf{b_1}$ pentagons, it has area $3\sqrt{25 + 10\sqrt{5}}\,\mathbf{b_1}^2$. (See Question 4 of Activity 15.1.) Observing that the distance between opposite edges is $\mathbf{b_3}$, you need to scale linearly by

$$\frac{1\text{ inch}}{\mathbf{b_3}}$$

which scales area by

$$\frac{1\text{ inch}^2}{(\tau^2\mathbf{b_1})^2}$$

So the dodecahedron has area

$$\frac{3}{\tau^4}\sqrt{25 + 10\sqrt{5}}\text{ inch}^2$$

numerically 3.01 inch^2. A sphere with radius $\frac{1}{2}$ has area π inch^2, so you can be confident you did not make a large mistake in this calculation.

Explorations 15

A. Each of the $2n$ right triangles has one leg $\frac{e}{2}$ opposite an angle of $\frac{180}{n}$. The other leg is then

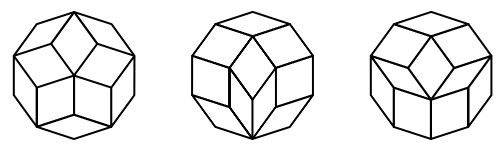
$$\frac{e}{2\tan\frac{180}{n}}$$, and the triangle's area is

$$\frac{e^2}{8\tan\frac{180}{n}}$$

The n-gon's area is then

$$\frac{ne^2}{4\tan\frac{180}{n}}$$

B. The figure below shows several rhombic dissections of the regular 10-gon. Every dissecting edge inside the 10-gon will be parallel to one of the 10-gon edges, because opposite edges of rhombi are parallel. So the rhombi angles must be the same as the angle between the edges of the 10-gon: 144 degrees; its supplement, 36 degrees; the angle remaining when 36 is subtracted from 144, 108 degrees; and its supplement, 72 degrees. Those are the only angles found in the rhombi. One method of proving that there must be five of each involves areas. Let x and y be their areas. If there are n of one and m of the other, then the area of the 10-gon must be $nx + my$. You know $m = n = 5$ gives solutions. Suppose there are other solutions, with i of one and j of the other, then $ix + jy$ gives the same area, so $nx + my = ix + jy$. You can solve this for the ratio $\frac{x}{y} = \frac{(j - m)}{(n - i)}$. The right side is a ratio of integers, so it is rational. But the left side, $\frac{x}{y}$, is irrational, which is a contradiction, so our supposition that there can be solutions other than 5 and 5 is incorrect. To see why $\frac{x}{y}$ is irrational, notice that each rhombus shape can be dissected into two of the isosceles triangles whose areas you determined in Question 3 of Activity 15.1.



Three rhombic dissections of the regular decagon

C. The intersection of the sphere and a midscribed polyhedron is a set of circles, one inscribed in each of the polyhedron's faces. The interiors of the circles lie inside the sphere. In Question 10 of Activity 15.2, you saw that the area of a dodecahedron midscribed to a sphere 1 inch in diameter is approximately 3.01 inch^2, which is about 4 percent less than the area of the sphere. You also saw (in Question 9 of 15.2) that a $\mathbf{b_1}$ icosahedron has area $5\sqrt{3}\mathbf{b_1}^2$. Its edge-to-edge diameter is $\mathbf{b_2}$. Scaling by

$$\left(\frac{1\text{ inch}}{\tau\mathbf{b_1}}\right)^2$$

shows that a 1-inch diameter icosahedron has an area of $5\frac{\sqrt{3}}{\tau^2}$ inch^2, which is approximately 3.31 inch^2, over 5 percent more than the area of the sphere.

16.1 Filling Space

Q1 Every edge is surrounded by three rhombic dodecahedra.

Q2 120 degrees

4. The elongated rhombic dodecahedron looks like a rhombic dodecahedron with four additional edges and four faces stretched into hexagons.

Q3 There are two types of edges. The new, vertical edges are surrounded by four elongated dodecahedra. Three elongated dodecahedra surround the slanted edges. Where four yellow edges meet, a pocket is formed into which a four-fold vertex of the next layer fits.

Q4 90 degrees at the vertical edges, 120 degrees at the slanted edges.

Q5 All edges are equivalent. A walk around an edge visits three truncated octahedra, passing through two hexagons and one square.

Q6 The octahedron's dihedral angle is 2γ, which remains unchanged as the hexagon-hexagon angle in the truncated octahedron. In the 360 degrees around an edge of the space structure, there is one of these angles and two of the other (square-hexagon) dihedral angle. So its value is

$$\frac{360 - 2\gamma}{2} = 180 - \gamma$$

7. **a.** Take one face of the cube and join its four vertices to the point at the center of the cube, to define a pyramid. Six of these make a cube, and copies of that fill space.

b. Either do as above with any rhombohedron, such as a pointy $\mathbf{b_2}$ rhombohedron (it will not be a right pyramid), or take one face of the rhombic dodecahedron and join its four vertices to a point at the center of the rhombic dodecahedron. Twelve of these make a rhombic dodecahedron, and copies of that fill space.

c. Take a square or rhombic dodecahedron pyramid above and slice it in half, cutting the base into two triangles. There are two ways of doing this with the rhombic dodecahedron pyramid, taking either the long or short diagonal.

d. Pick one of the square or rhombic pyramids above and put two back-to-back.

e. There are many possibilities. For example, assemble two of the six pyramids that form a cube (joining a triangle to a triangle). Or assemble three of them together, either in a U shape or each in contact with the other two.

Q7 Circling any edge, you pass through an octahedron, a tetrahedron, another octahedron, and another tetrahedron.

Q8 Octahedron: 2γ, approximately 109.5 degrees. Tetrahedron: $180 - 2\gamma$, approximately 70.5 degrees.

16.2 Packing Spheres

Challenge Cannonball stacks (where each touches 12 neighbors) are denser than balls stacked in a cubical array (where each touches 6 neighbors).

Q1 Each ball touches 12 neighbors, 4 in its layer, 4 in the layer above, and 4 in the layer below.

Q2 Four planes of triangular arrangement are easy to see, being parallel to the pyramid's triangular sides. Two other planes of squares are less obvious, but notice that the bottom center ball, the top ball, and two of the balls of the middle layer together form a square in a vertical plane. There are two such planes, depending on which two balls of the middle layer are chosen. The three planes of squares are mutually perpendicular.

Q3 Again, each ball touches 12 neighbors, now 6 in its layer, 3 in the layer above, and 3 in the layer below.

Q4 The triangles in horizontal planes are obvious. Three other planes of triangular arrangement are also easy to see, being parallel to the pyramid's triangular sides. Three planes of squares are less obvious, but take off the top ball, one of the three under it, and one under that, to see two squares in a plane. There are three such planes, depending on which ball of the second layer is chosen.

Q5 If replicated infinitely, the two stacks of cannonballs have the identical structure; they are just oriented differently with respect to gravity and sliced along different planes.

Q6 A cuboctahedron

Q7 They are the vertices of an *isomer* of the cuboctahedron. The polyhedron is formed by cutting a cuboctahedron along a hexagon equator and putting the halves back together with a 60-degree twist so that along the equator squares border squares and triangles border triangles.

16.3 The FCC Lattice

Challenge This gives the face-centered-cubic sphere packing.

Q1 The two structures have the same edges. You are just visually grouping an octahedron with four surrounding tetrahedra into a stella octangula.

Q2 The second layer of squares is the dual tessellation to the first. Looking down, you see how each edge of one crosses an edge of the other. In addition to square pyramids, there are tetrahedra. Crossed pairs of edges in the two tessellations contribute two edges to each tetrahedron—one in each plane. The other four edges come from square pyramids.

Q3 Two square pyramids join to form an octahedron. The result is the space structure of alternating octahedra and tetrahedra (compressed slightly, to be not regular).

5. The "squares" in vertical planes are compressed into yellow rhombi.

Q4 Octahedra and tetrahedra

Explorations 16

A. The result is again alternating tetrahedra and octahedra, but possibly very stretched or compressed along a three-fold axis.

B. An octahedron has twice as many faces as a tetrahedron, and each face of the space structure borders one of each, so there must be twice as many tetrahedra. The same conclusion results from the fact that an octahedron has twice as many edges as a tetrahedron and each edge of the space structure contacts two of each. A third method is to count that each vertex of the space structure is shared by six octahedra and eight tetrahedra. The octahedra have six vertices, so with n octahedra you count $6n$ vertices, but each one was counted six times; so correct for the overcount by dividing by six, finding that there is one vertex per octahedron on average. The tetrahedra have four vertices, so there is only one-half vertex per tetrahedron on average.

C. The concave dodecahedron and the regular dodecahedron can fill space together. Imagine cubes filling space colored alternately black and white. Put a dodecahedron around all the white ones, and then take six roof shapes away from the black ones.

D. Use four 5-gons, two 4-gons, and eight 3-gons, all regular, in a blue structure. The squares are surrounded by triangles but at two different dihedral angles.

E. See Robert Williams, *The Geometrical Foundation of Natural Structure,* for pictures of these structures.

F. This structure can be formed in a blue cube, using alternating vertices of the cube. It can be seen as four different planes of skew hexagons. It is half the edges of the packing of rhombic dodecahedra. It can also be seen as putting four yellow struts inside alternating tetrahedra in the tetrahedron and octahedron structure.

G. The regular dodecahedron's dihedral angle is approximately 116.6 degrees, not 120. So, if three regular dodecahedra meet at a common edge, there will be a small wedge of space unfilled. This gap can be closed by a slight rotation through the fourth dimension, as seen in Activity 21.3!

H. In each case, the pieces fit together as in the space structure of alternating tetrahedra and octahedra. Each of the shapes can be found among the edges of the structure.

I. Each low pyramid is one fourth of a regular tetrahedron. When the g_1-y_1-y_1 isosceles triangles are joined with others, regular tetrahedra are formed that alternate with the truncated tetrahedra. So this polyhedron originates with a structure of alternating tetrahedra and truncated tetrahedra, then grows. The tetrahedra are divided into pyramids that meet at its center, and the pieces are attached to the other polyhedra they touch. Applying this same operation to the structure of alternating octahedra and tetrahedra gives the rhombic dodecahedron if the tetrahedra are divided or gives the cube if the octahedra are divided.

J. The space structure consists of pyramids (from the polygons) and tetrahedra (from the crossing edges). The back-to-back pyramids can be joined into dipyramids.

17.1 Prisms and Scaling

Challenge If you find one smaller, let us know. Make a b_1 rhombus in the red plane, having a 36-degree vertex angle. Inserting a red strut perpendicular to the rhombus makes it easy to see that there are 10 yellow holes in the zomeballs, which are only a small angle from the red plane. Pick any one of those 10 directions and place a y_1 in each ball, to locate the other four vertices. From Unit 13, that red-yellow angle is $90 - \alpha + \beta$ (or its supplement $90 + \alpha - \beta$), so the angle between the yellow strut and the red plane is $\beta - \alpha$. The height of the prism is $\sin(\beta - \alpha)y_1$.

Smaller one by Bob Mead: A low slanted triangular prism built on a b_1-y_1-y_1 triangle, using either four b_1s or four y_1s for the side edges. The height is the same in either case, and the volume is $0.0773b_1^3$.

1. The golden brick has edges $\mathbf{b_1}$, $\mathbf{b_2}$, and $\mathbf{b_3}$. Any pair can form the base, and the third becomes the height. Its volume is $\mathbf{b_1b_2b_3} = \tau^3\mathbf{b_1}^3$. This is the same volume as a $\mathbf{b_2}$ cube.

Q1–Q2 In models with edge $\mathbf{2r_1}$, the acute rhombohedron has height $h = \mathbf{b_2}$ and the obtuse rhombohedron has $h = \mathbf{b_1}$. Because they have the same base, this shows the ratio of their volumes is τ. The $\mathbf{r_1}$ rhombohedra have half these heights, respectively. You saw in Unit 15 that the base has area $\frac{\tau}{2}\mathbf{b_1}^2$, so the volumes are $\frac{\tau^2}{4}\mathbf{b_1}^3$ and $\frac{\tau}{4}\mathbf{b_1}^3$, respectively. Thus the ratio of their volumes is τ.

Q3 The $\mathbf{r_1}$ rhombic dodecahedron is the sum of two acute and two obtuse rhombohedra, so its volume is

$$2\left(\frac{\tau^2}{4}\right)\mathbf{b_1}^3 + 2\left(\frac{\tau}{4}\right)\mathbf{b_1}^3 = (\tau + 1)\frac{\tau}{2}\mathbf{b_1}^3 = \frac{\tau^3}{2}\mathbf{b_1}^3$$

which is half a $\mathbf{b_2}$ cube.

Q4 **a.** 4 or 2^2

b. 8 or 2^3

Q5 **a.** For the smaller prism, $A = 2(lw + wh + hl)$.
For the larger prism,
$A' = 2(2l \cdot 2w + 2w \cdot 2h + 2h \cdot 2l)$
$= 4 \cdot 2(lw + wh + hl) = 4A.$

b. For the smaller prism, $V = lwh$.
For the larger prism, $V' = 2l \cdot 2w \cdot 2h = 8lwh = 8V.$

Q6 **a.** Smaller: $A = 2(lw + wh + hl)$
Larger: $A' = 2(kl \cdot kw + kw \cdot kh + kh \cdot kl) =$
$k^2 \cdot 2(lw + wh + hl) = k^2A$

b. Smaller: $V = lwh$
Larger: $V' = kl \cdot kw \cdot kh = k^3lwh = k^3V$

Q7 The $\mathbf{r_3}$ rhombic dodecahedron is scaled up by a linear dimension of τ^2, therefore its volume is scaled by $(\tau^2)^3$, which is τ^6, making its volume $\frac{\tau^9}{2}\mathbf{b_1}^3$.

Q8 This is numerically about 38 $\mathbf{b_1}$ cubes, equivalent to a $\mathbf{b_1}$-by-$\mathbf{b_1}$-by-38 $\mathbf{b_1}$ prism.

17.2 Pyramids and Beyond

Challenge Make a $\mathbf{b_2}$ decagon, and extend every other edge with $\mathbf{b_1}$ struts in both directions. They connect to form a pentagon. Radial struts that locate the center of the decagon also locate the center of the pentagon.

Q1 The only symmetric pyramid on a $\mathbf{b_1}$ square base has $\mathbf{y_1}$ struts for its slanted edges. It has height $h = \frac{1}{2}\mathbf{b_1}$, which can be seen either from a double-scale model or by noticing that two of these pyramids facing in from opposite sides of a cube will meet at the cube's center. Applying the formula one-third base times height gives the volume as $\frac{1}{6}\mathbf{b_1}^3$. This result can be seen more directly by noticing that a $\mathbf{b_1}$ cube can be dissected into six of these pyramids meeting at the cube's center, so each has one sixth the cube's volume.

Q2 Since the rhombic dodecahedron adds $6 \cdot \left(\frac{1}{6}\right)\mathbf{b_1}^3$ to a $\mathbf{b_1}$ cube, its total volume is $\mathbf{2b_1}^3$.

Q3 A $\mathbf{3b_1}$ equilateral triangle allows you to add interior $\mathbf{b_1}$ struts to locate its center. Building the $\mathbf{3r_1}$ slanting edges, you can determine from the model that $h = \mathbf{y_3}$. Scaling down by a factor of three, the $\mathbf{b_1}$ pyramid has $h = \frac{\mathbf{y_3}}{3} = \frac{\tau^2\sqrt{3}}{6}\mathbf{b_1}$. The area of a $\mathbf{b_1}$ equilateral triangle is $\frac{\sqrt{3}}{4}\mathbf{b_1}^2$. So the pyramid volume is $\frac{1}{3}\frac{\sqrt{3}}{4}\mathbf{b_1}^2\ \frac{\tau^2\sqrt{3}}{6}\mathbf{b_1} = \frac{\tau^2}{24}\mathbf{b_1}^3$.

Q4 A $\mathbf{b_1}$ icosahedron can be dissected into 20 of the above pyramids (recall the red starburst), so its volume is $\frac{5\tau^2}{6}\mathbf{b_1}^3$. Scaling $\mathbf{b_1}$ up to 1 meter, an icosahedron with edge 1 meter has volume $\frac{5\tau^2}{6}$ meter^3, numerically 2.18 meter^3.

Q5 A dodecahedron can be dissected into 12 pentagonal pyramids that meet at the dodecahedron's center. This is illustrated by the yellow starburst method of its construction. For a $\mathbf{b_1}$ dodecahedron, the pentagonal pyramid has a $\mathbf{b_1}$ base and $\mathbf{y_2}$ slanting edges.

Q6 To find the pyramid's height, you first need to locate a pentagon's center. This was solved in the Challenge. That pentagon has edge $\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_1} = (2 + \tau)\mathbf{b_1}$, so the similarly scaled yellow edges have length $(2 + \tau)\mathbf{y_2} = \mathbf{y_2} + \mathbf{y_3} + \mathbf{y_2}$. Making a model shows that

$$h = \mathbf{r_3} + \mathbf{r_2} = (\tau + \tau^2)\mathbf{r_1} = (1 + 2\tau)\mathbf{r_1}$$
$$= (1 + 2\tau)\frac{\sqrt{2 + \tau}}{2}\mathbf{b_1}$$

where the last step uses the fact, shown in Unit 13, that $\mathbf{r_1} = \frac{\sqrt{2 + \tau}}{2}\mathbf{b_1}$. To scale this down to the $\mathbf{b_1}$ size base, linear dimensions are divided by $(2 + \tau)$, and

$$h = (1 + 2\tau)\frac{\sqrt{2+\tau}}{2(2+\tau)}\mathbf{b_1}$$

Recalling, from Unit 15, that the $\mathbf{b_1}$ pentagon has area

$$\frac{\sqrt{25+10\sqrt{5}}}{4}\mathbf{b_1}^2$$

you multiply one-third base times height to obtain a rather messy formula for the pyramid's volume:

$$\frac{(\sqrt{25+10\sqrt{5}})(1+2\tau)(\sqrt{2+\tau})}{24(2+\tau)}\mathbf{b_1}^3$$

With work, this can be simplified analytically to

$$\frac{15+7\sqrt{5}}{48}\mathbf{b_1}^3$$

Q7 Sum 12 of the above pyramids to get

$$\frac{15+7\sqrt{5}}{4}\mathbf{b_1}^3$$

This can also be expressed as

$$\frac{\sqrt{5}\tau^4}{2}\mathbf{b_1}^3$$

Numerically, it is 7.66 $\mathbf{b_1}^3$.

Explorations 17

A. Volume $= 5\sqrt{\frac{47+21\sqrt{5}}{40}}x^3$. Numerically, this is also $7.66x^3$.

B. If A_3 shrinks to a point, the shape is a pyramid and A_2 is the cross section halfway up. A_2 is then similar to A_1, but with half the linear dimensions, so it is a quarter of the area. The formula $\frac{(A_1 + 4A_2 + A_3)h}{6}$, after substituting $A_3 = 0$ and $A_2 = \frac{A_1}{4}$, reduces to $\frac{A_1 h}{3}$, the volume of a pyramid.

C. A thirtieth of an $\mathbf{r_1}$ rhombic triacontahedron (RT) is a pyramid with $\mathbf{r_2}$ and $\mathbf{y_2}$ slanting edges. In double scale, you see a $\mathbf{2r_1}$ version has height $\mathbf{b_3}$. The $\mathbf{r_1}$ pyramid has base area $\frac{\tau}{2}\mathbf{b_1}^2$, height $\frac{\tau^2}{2}\mathbf{b_1}$, and volume $\frac{1}{12}\tau^3\mathbf{b_1}^3$. Thirty of them give the total RT volume as $5\frac{\tau^3}{2}\mathbf{b_1}^3$. Summing the 20 rhombohedra gives the same result: $10\frac{\tau^2}{4}\mathbf{b_1}^3 + 10\frac{\tau}{4}\mathbf{b_1}^3 = 5\frac{\tau^3}{2}\mathbf{b_1}^3$

18.1 Zonish Big Domes

Q1 It takes $n - 3$ diagonals to triangulate an n-gon. The more sides a polygon has, the more choices there are as to how to arrange these diagonals.

Q2 There are 12 pentagons and 30 irregular hexagons. The form can be obtained by truncating just the five-fold vertices of the rhombic triacontahedron. Each hexagon is the central part of a red rhombus, after its two acute vertices are truncated.

Explorations 18

B. The result has 12 pentagons, 60 rectangles, 60 rhombi, and 30 skinny hexagons.

C. Between the blue triangles and pentagons will be yellow zones consisting of irregular hexagons and skinny rhombi. In size $\mathbf{y_3}$, you can brace the rhombi with $\mathbf{b_2}$s as their short diagonals.

D. The result has 8 triangles, 30 rectangles, and 12 fat yellow rhombi (arranged as in the rhombic dodecahedron). It appears in two books by Peter Pearce: *Structure in Nature Is a Strategy for Design* (p. 49) and *Polyhedra Primer* (p. 50), but not with this method of construction.

19.1 Vertex Coordinates

Challenge The three adjacent nodes are at distance $\mathbf{2b_1}$. After that, the next closest six nodes are at distance $\mathbf{2b_2}$ (the diagonal of a $\mathbf{2b_1}$ pentagon). To find the distance to the next six nodes, visualize the $\mathbf{2b_2}$ cube in the $\mathbf{2b_1}$ dodecahedron. You can see this is the distance of a diagonal of a face of that cube, so it is $2\sqrt{2}\mathbf{b_2} = 2\sqrt{2}\tau\mathbf{b_1}$, because the diagonal of a square is $\sqrt{2}$ times its edge. Finally, there are three further nodes at distance $\mathbf{2b_3}$ and one node—the opposite vertex—at distance $\mathbf{4y_2}$. Cartesian coordinates provide another approach to this problem in Questions 9 and 10.

Q1 The coordinates of the vertices are (1, 1, 1), (1, 1, –1), (1, –1, 1), (1, –1, –1), (–1, 1, 1), (–1, 1, –1), (–1, –1, 1), and (–1, –1, –1).

Q2 We could scale down to half the size and obtain an edge length of 1 if the coordinates were $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$.

Q3 One solution is (1, 1, 1), (1, –1, –1), (–1, 1, –1), (–1, –1, 1), where the +1 always appears an odd number of times. The other solution is the set of cube vertices where +1 appears an even number of times.

Q4 It is a distance $\mathbf{b_1}$ in the negative x-direction. It is at a position 0 along the y-axis. It is a distance $\mathbf{b_2}$ in the positive z-direction. So its coordinates might be written $(-\mathbf{b_1}, 0, \mathbf{b_2})$.

Q5 $(-1, 0, \tau)$ with units $\mathbf{b_1}$ understood

Q6 The icosahedron's 12 vertices are naturally grouped into three sets of four: $(\pm1, 0, \pm\tau)$, $(0, \pm\tau, \pm1)$, $(\pm\tau, \pm1, 0)$. Note how the components shift over one position.

Q7 The first edge has endpoints $(\tau, 1, 0)$ and $(\tau, -1, 0)$. The length of the edge is $\sqrt{(\tau-\tau)^2 + (1-(-1))^2 + (0-0)^2} = 2$. This is two units of $\mathbf{b_1}$, the length we chose for the model. The length of the edge from $(1, 0, \tau)$ to $(0, \tau, 1)$ should work out to be the same. It is

$$\sqrt{(1-0)^2 + (0-\tau)^2 + (\tau-1)^2}$$

$$= \sqrt{1 + \tau^2 + (\tau^2 - 2\tau + 1)}$$

$$= \sqrt{2 - 2\tau + 2\tau^2}$$

$$= \sqrt{2(1 + (\tau^2 - \tau))} = \sqrt{4} = 2, \text{ again.}$$

Q8 In this position, the vertex coordinates are $(\pm1, \pm\tau, 0)$, $(\pm\tau, 0, \pm1)$, and $(0, \pm1, \pm\tau)$.

Q9 Assuming units of $\mathbf{b_1}$ again, the coordinates of the dodecahedron vertices are $(\pm\tau^2, 0, \pm1)$, $(\pm1, \pm\tau^2, 0)$, $(0, \pm1, \pm\tau^2)$, and $(\pm\tau, \pm\tau, \pm\tau)$. The last group is the eight vertices of one of the cubes in the dodecahedron.

Q10 The distances are those given in the answer to the Challenge.

19.2 Point Operations

Challenge Five examples are given below: the cuboctahedron (3, 4, 3, 4) is Question 5; the icosidodecahedron (3, 5, 3, 5) is Question 6; the truncated icosahedron (5, 6, 6) is Question 11; the truncated tetrahedron (3, 6, 6) is Question 12; the truncated cube (3, 8, 8) is Exploration G.

Q1 $\sqrt{3}$

Q2 For the eight points of the form $p = (\pm\tau, \pm\tau, \pm\tau)$, $|p| = \sqrt{\tau^2 + \tau^2 + \tau^2} = \sqrt{3}\tau$. For the twelve that are permutations of $p = (0, \pm1, \pm\tau^2)$, $|p| = \sqrt{1 + \tau^4} = \sqrt{1 + 2 + 3\tau} = \sqrt{3(1+\tau)} = \sqrt{3}\tau$. (Use the τ manipulations from Unit 13.) The changes in order or sign of the components do not affect the squared sum. So every vertex has magnitude $\sqrt{3}\tau$, which is $\mathbf{2y_2}$ in units of $\mathbf{b_1}$, which we know is correct from a yellow starburst.

Q3 For vertices, p, of the $\mathbf{2b_1}$ icosahedron, $|p| = |(\pm1, 0, \pm\tau)| = \sqrt{1^2 + 0^2 + \tau^2} = \sqrt{1 + \tau^2} = \sqrt{2 + \tau}$. Scaling by $\frac{1}{\sqrt{2+\tau}}$ will give points a distance 1 from the origin, so four of the vertices are

$$\left(\frac{\pm1}{\sqrt{2+\tau}}, 0, \frac{\pm\tau}{\sqrt{2+\tau}}\right)$$

The others come from rotating the three coordinates as in Question 6 of Activity 19.1.

1. It is a $\mathbf{2r_1}$ rhombus. The top edge of the icosahedron is its short diagonal.

Q4 Directly above the origin, halfway to the point $p_1 + p_2$, is the center of the red rhombus—the midpoint of the segment connecting p_1 and p_2.

Q5 $\frac{1}{2}(p_1 + p_2) = \frac{1}{2}(2, 2, 0) = (1, 1, 0)$

Q6 The 12 vertices are $(\pm1, \pm1, 0)$, $(\pm1, 0, \pm1)$, and $(0, \pm1, \pm1)$.

Q7 There are six icosidodecahedron vertices on the axes: $(\pm\tau, 0, 0)$, $(0, \pm\tau, 0)$, and $(0, 0, \pm\tau)$. (These are the vertices of a regular octahedron.) A typical off-axis midpoint is $\left(\frac{\tau^2}{2}, \frac{1}{2}, \frac{\tau}{2}\right)$, which is the midpoint of the segment joining $(1, 0, \tau)$ and $(\tau, 1, 0)$. The 24 off-axis vertices come in three groups of eight by changing signs and shifting the coordinates: $\left(\frac{\pm\tau^2}{2}, \frac{\pm1}{2}, \frac{\pm\tau}{2}\right)$, $\left(\frac{\pm1}{2}, \frac{\pm\tau}{2}, \frac{\pm\tau^2}{2}\right)$, and $\left(\frac{\pm\tau}{2}, \frac{\pm\tau^2}{2}, \frac{\pm1}{2}\right)$. Each group of eight defines a golden brick.

Q8 $(1, 0, 0)$

Q9 The six vertices are $(\pm1, 0, 0)$, $(0, \pm1, 0)$, and $(0, 0, \pm1)$.

Q10 The first face has vertices $(\tau, 1, 0)$, $(\tau, -1, 0)$, and $(1, 0, \tau)$. The face center is their average, $\left(\frac{1+2\tau}{3}, 0, \frac{\tau}{3}\right)$. Note that it is on the xz-plane. The second face has vertices $(\tau, 1, 0)$, $(1, 0, \tau)$, and

$(0, \tau, 1)$. The center is their average, $\left(\frac{\tau^2}{3}, \frac{\tau^2}{3}, \frac{\tau^2}{3}\right)$. Note that it is on the line $x = y = z$.

Q11 $\frac{3}{\tau}$

Q12 $p_1 = (-1, 0, \tau)$ and $p_2 = (+1, 0, \tau)$, so the points that trisect it are $\frac{1}{3}p_1 + \frac{2}{3}p_2 = \left(+\frac{1}{3}, 0, \tau\right)$ and $\frac{1}{3}p_2 + \frac{2}{3}p_1 = \left(-\frac{1}{3}, 0, \tau\right)$.

Q13 The tetrahedron vertices in the **2b$_1$** cube are $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$. The trisection points are $\left(\frac{1}{3}, \frac{1}{3}, 1\right), \left(-\frac{1}{3}, -\frac{1}{3}, 1\right), \left(-1, \frac{1}{3}, -\frac{1}{3}\right)$, and so on, that is, the 12 permutations of $\left(\pm 1, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$, which have an odd number of positive entries. For the **3b$_1$** cube, scale by $\frac{3}{2}$.

Explorations 19

A. Rotating about the z-axis, the new coordinates are $(\pm\sqrt{2}, 0, \pm 1)$ and $(0, \pm\sqrt{2}, \pm 1)$.

B. It is the line $x = y = z$, since points of the form (x, x, x) remain fixed by the operation.

C. The result is a dodecahedron with a pyramid of equilateral triangles inside each face and an inner icosahedron. Removing the edges of the icosahedron leaves the *concave equilateral deltahedron* seen in Exploration D in Unit 2. Removing instead the outer dodecahedron leaves the great stellated dodecahedron to be seen again in Unit 20.

D. The result has one edge and one diagonal in each face. It rests on any of the pentagonal sides. There are two ways to start, that is, $(\pm 1, \pm\tau^2, 0)$ or $(\pm\tau^2, \pm 1, 0)$ for the first rectangle. In one case the dodecahedron can be completed; in the other it can't.

E. Let four vertices be $O = (0, 0, 0)$, $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, and $C = (x_3, y_3, z_3)$. The other four vertices are $A + B$, $A + C$, $B + C$, and $A + B + C$. The sum of the squares of the lengths of the edges is $4(|A|^2 + |B|^2 + |C|^2) = 4(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 + z_1^2 + z_2^2 + z_3^2)$. The sum of the squares of the lengths of the diagonals is $|A + B + C|^2 + |A + B - C|^2 + |A - B + C|^2 + |-A + B + C|^2$, which when expressed in terms of the x's, y's, and z's simplifies to the same expression.

F. This depends on available software.

G. The one-third points produce regular hexagons from equilateral triangles but do not produce regular octagons from squares. Consider the ruler-and-compass construction of the regular octagon. Draw the diagonals in a square. A compass centered on a vertex with half the diagonal as radius marks two of the octagon's vertices on the squares. From this, the correct weighting is $\frac{\sqrt{2}}{2}p_1 + \left(1 - \frac{\sqrt{2}}{2}\right)p_2$. The 24 vertices are the permutations of $(\pm 1, \pm 1, \pm\sqrt{2} - 1)$.

H. One edge has endpoints $p = \left(\pm\frac{1}{3}, 0, \tau\right)$. The edge length is $\frac{2}{3}$ and the sphere radius is $|p| = \sqrt{\frac{1}{9} + \tau^2}$. The ratio of radius to edge is approximately 2.47.

20.1 Self-Intersecting Polygons

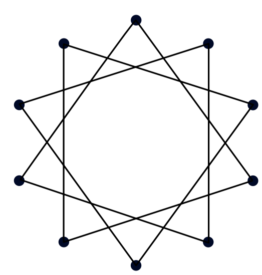
1. Start with a pentagon of five **b$_1$**s. Extend each side in both directions by adding ten **b$_2$**s. Five more balls connect everything together.

Q1 Five edges, five vertices

Q2 Two concentric pentagons cross through each other. Each has edge **b$_1$** + **b$_2$** + **b$_1$**. Each edge intersects two edges of the other pentagon.

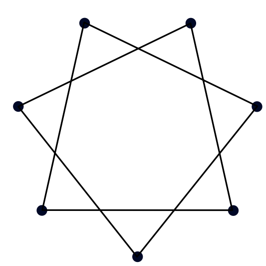
Q3 The result is a self-intersecting regular 10-gon.

Q4

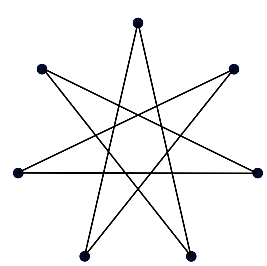


$\frac{10}{3}$-gon

Q5



$\frac{7}{2}$-gon



$\frac{7}{3}$-gon

Q6 A $\frac{7}{1}$-gon is the standard regular 7-gon. Any $\frac{n}{1}$-gon is the same as a standard regular n-gon.

Q7 A pentagram is a $\frac{5}{2}$-gon.

Q8 A $\frac{10}{7}$-gon is the same as a $\frac{10}{3}$-gon, drawn in reverse.

Q9 If $m + k = n$, then an $\frac{n}{m}$-gon is identical to an $\frac{n}{k}$-gon, because going around m hops in one direction is equivalent to going around k hops in the other direction. Equivalently, an $\frac{n}{m}$-gon is identical to an $\frac{n}{(n-m)}$-gon.

Q10 Using the preceding procedure to draw an $\frac{n}{m}$-gon where m divides evenly into n, or where n and m have a common factor, some of the n vertices will not be visited.

Q11 $180 - \frac{360m}{n}$. This follows from the fact that a turtle walking along the sides would make n turns, adding up to a total turning of m full turns. Since the total turning is $360m$, each turn is $\frac{360m}{n}$, and the interior angle is the supplement of that.

20.2 The Kepler-Poinsot Polyhedra

Challenge A prism based on the $\frac{10}{3}$-gon is one answer. The Kepler-Poinsot polyhedra are others.

Q1 There are 12 regular pentagons, 2 per edge, 5 per vertex. There are still 30 edges.

2. In type (b), *exterior lines of intersection,* there is a triangular pyramid constructed inside each of the icosahedron's faces, using $\mathbf{b_2}$ struts. In type (c), *all lines of intersection,* there is a complete pentagram inside each of the 12 pentagons.

Q2 $V = 12$, $F = 12$, and $E = 30$; so $V + F \neq E + 2$. Without even adding, you might notice that V and E are the same as for the icosahedron, but F changes from 20 to 12, so the equality cannot still hold. The great dodecahedron does not satisfy the conditions of Euler's theorem; it is not a convex, simply connected polyhedron.

Q3 It can be seen as a self-intersecting polyhedron with 12 pentagram faces, 30 edges, and 12 vertices. The $\mathbf{b_2} + \mathbf{b_1} + \mathbf{b_2}$ edges are of length $\mathbf{b_4}$, and two faces meet along each edge as required.

Q4 At each vertex of the small stellated dodecahedron, any two nonadjacent edges meet at a 60-degree angle. Examining the model shows that there are 20 equilateral triangles, each centered on a three-fold axis of symmetry. Two triangular faces meet per edge. There are still 12 vertices and 30 edges, as in the small stellated dodecahedron.

Q5 There are 12 pentagrams, meeting 3 to a vertex. There are 20 vertices and 30 edges.

Q6 The number of faces of X equals the number of vertices of Y and vice versa.

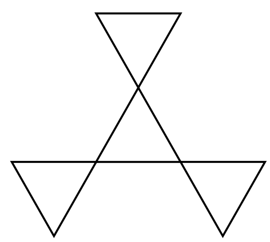
Q7 The great dodecahedron ($V = 12$, $F = 12$) and small stellated dodecahedron ($V = 12$, $F = 12$) are a dual pair. The great icosahedron ($V = 12$, $F = 20$) and great stellated dodecahedron ($V = 20$, $F = 12$) are another dual pair.

Q8 The small stellated dodecahedron is $\left\{\frac{5}{2}, 5\right\}$, since five pentagrams meet at each vertex. The great stellated dodecahedron is $\left\{\frac{5}{2}, 3\right\}$, since three pentagrams meet at each vertex.

Q9 The great dodecahedron is $\left\{5, \frac{5}{2}\right\}$, since it is dual to the small stellated dodecahedron, $\left\{\frac{5}{2}, 5\right\}$. The great icosahedron is $\left\{3, \frac{5}{2}\right\}$, since it is dual to the great stellated dodecahedron, $\left\{\frac{5}{2}, 3\right\}$.

20.3 Uniform Polyhedra

Challenge



A uniform 6-gon

Q1 Only the outer six balls correspond to vertices; the other balls are just for the construction. The edges may be several struts in length.

Q2 The regular 10-gons found at the equators of the icosidodecahedron are one type of face in our new self-intersecting polyhedra. Each edge is part of exactly one 10-gon, so we need to find another polygon to be the second polygon at each edge. One way to do this is to keep all the triangles of the icosidodecahedron, which gives us a polyhedron (3, 10, 3, 10) with six 10-gons and twenty 3-gons. In the pentagonal holes, one can see parts of the 10-gons. Notice that the two 10-gons that meet at a vertex cross each other, so when you go around a vertex, you actually make a figure 8.

Q3 For the second self-intersecting polyhedron, we use the 10-gons and take just the pentagons of the icosidodecahedron. This gives (5, 10, 5, 10) with six 10-gons and twelve 5-gons. Parts of the 10-gons can be seen through the triangular holes. Note that this polyhedron does not have a single inside but, rather, can be seen as a combination of pyramids, which share a common apex.

Q4 Regular 10-gons are found among the edges of the rhombicosidodecahedron. They are one type of face in our new self-intersecting polyhedra. Notice that each rhombicosidodecahedron edge is part of exactly one 10-gon, so we need to find another polygon to be the second polygon at each edge. One way to do this is to keep all the squares of the rhombicosidodecahedron, which gives us a polyhedron (4, 10, 4, 10) consisting of twelve 10-gons and thirty 4-gons. Parts of the 10-gons can be seen through the triangular and pentagonal holes. For the second self-intersecting polyhedron, we keep the triangles and pentagons of the rhombicosidodecahedron, but throw out the squares. This gives (3, 10, 5, 10) with twelve 10-gons, twelve 5-gons, and twenty 3-gons. Parts of the 10-gons can be seen through the square holes.

Explorations 20

A. A yellow one starts with a regular skew 10-gon from $\mathbf{y_1}$s in which each angle is the same as the acute angle of a fat yellow rhombus. It is extended with $\mathbf{y_2}$s in each direction to make a taller, narrower crown. Each is the zig-zag of a pentagrammatic crossed antiprism with isosceles side faces.

B. The compound of the small stellated dodecahedron with the great dodecahedron starts as the compound of the $\mathbf{2b_1}$ dodecahedron with the $\mathbf{2b_2}$ icosahedron (see Unit 9). Each dodecahedron face is then elevated with $\mathbf{2b_2}$ slanting edges to make the small stellated dodecahedron. The compound of the great stellated dodecahedron with the great icosahedron extends this by adding triangular pyramids of length $\mathbf{2b_3}$ to all the icosahedron faces. However, in order to cross the midpoints of the pentagonal pyramids, the $\mathbf{2b_3}$ lengths are constructed as $\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}$.

C. The compound has the same edges as uniform polyhedra with vertices of types (3, 5, 3, 5, 3, 5), $\left(3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}\right)$, and $\left(5, \frac{5}{2}, 5, \frac{5}{2}, 5, \frac{5}{2}\right)$.

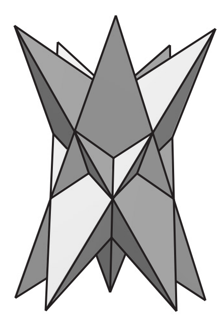
D. **a.** The result is $\left(\frac{5}{2}, 10, 10\right)$ with twelve pentagrams and twelve 10-gons.

b. The result is $\left(\frac{5}{2}, 5, \frac{5}{2}, 5\right)$ with twelve pentagrams and twelve pentagons. It has the same vertices as the icosidodecahedron.

c. It has the vertices of the icosidodecahedron, but with six equatorial $\frac{10}{3}$-gons rather than 10-gons.

E. The octahedron has three square equators. If we make a polyhedron using those squares and alternating triangular faces of the octahedron, we have a polyhedron of type (3, 4, 3, 4) with three squares and four triangles. The squares pass through each other (and the center of the polyhedron), so it is not as peculiar as the Möbius strip, but tracing over the surface like an ant, you can verify that it has only one side.

F. The figure shows how the ten arrowhead faces connect. Notice there are two per edge, as there should be.



Pentagrammatic concave trapezohedron

G. The result is the same as building a $\mathbf{b_1}$ icosidodecahedron, erecting two pyramids (with slanting edges of $\mathbf{y_1}$ and $\mathbf{y_2}$) on each pentagon, then removing the $\mathbf{b_1}$s. There are 12 five-fold vertices and 12 vertices, so it is dual to the $\left(5, \frac{5}{2}, 5, \frac{5}{2}\right)$. It can also be seen as a stellation of the rhombic triacontahedron. (See Activity 22.1 for the definitions of stellation.)

21.1 Hypercubes

Challenge One way to do this is to start with a cube, then add a zone in a red or yellow direction. See Question 9.

Q1 Each is double the previous (because we connected two copies of the previous hypercube). Generally, we expect 2^n vertices in an n-dimensional hypercube.

Q2 The 4D hypercube should have $2^4 = 16$ vertices.

Q3 Each hypercube has double the number of edges in the previous (because we connected two copies of the previous hypercube) plus the number of vertices in the previous (because we added a new edge for each vertex of the previous hypercube). So the 4D hypercube should have $2(12) + 8 = 32$ edges. Generally, from the fact that each vertex touches n edges, there are $n\frac{2^n}{2} = n(2^{n-1})$ edges.

Q4 The figure follows the pattern, with 16 vertices and 32 edges. There are two copies of the 3D cube, and their corresponding vertices are connected with yellow struts. Since there is no new direction that is perpendicular to the previous three, we use *inward* or *outward* as the new direction.

Q5 Like the first drawing, the inner and outer components are undistorted and different sizes, but angles are distorted in the connecting components. It is centered on the center of a cube (the hypercube one dimension down) just as the first cube drawing is centered on the center of a square.

Q6 The figure contains 16 vertices and 32 edges, as expected. Hold a three-fold axis vertical to see that there are two copies of a rhombohedron, one high and one low, with their corresponding vertices connected by vertical struts.

Q7 Like the second drawing, squares appear as rhombi and the model is centered on two opposite vertices that are slightly displaced, both trying to be at the center.

Q8 The two components of lower dimension are undistorted and the same size, but the connecting lines are not perpendicular to the cubes.

Q9 The outer shell of the model answers the Challenge.

Q10 A 4D hypercube is bounded by eight cubes. The eight cubes can be found in all three models, although most are distorted.

Q11 Model 5 shows the eight component cubes of the 4D hypercube, each undistorted. But it has been separated, along faces, which need to be rejoined. Because it has been unfolded, it does not make clear that only opposite pairs of cells are actually parallel.

21.2 Simplexes

Challenge The most symmetric way to do five balls is to build a green regular tetrahedron and then join all vertices to a center zomeball with yellow struts. For six or seven balls, see Exploration B.

Q1 Each simplex has one more vertex than the previous. Generally, there are $n + 1$ vertices in an n-dimensional simplex.

Q2 Five vertices

Q3 Each simplex has the number of edges in the previous plus the number of vertices in the previous (because we add a new edge for each vertex of the previous simplex). Because every pair of vertices is connected by an edge, there are $\frac{n(n+1)}{2}$ edges in an n-dimensional simplex.

Q4 The 4D simplex should have 6 + 4, or $\frac{4 \cdot 5}{2}$, = 10 edges.

Q5 It starts as a tetrahedron, to which is added a new vertex that connects with all the tetrahedron vertices. It has five vertices and ten edges as expected.

Q6 A 4D simplex is bounded by five tetrahedra. These five cells can be seen directly in the model. One is the exterior tetrahedron.

Q7 A pentagon and pentagram superimposed (sharing the same five vertices)

Q8 Given an edge, there are three edges that do not share a vertex with it (and so can be opposite it in a tetrahedron) and therefore three tetrahedra including it. That would give 10 times 3 tetrahedra, but since each tetrahedron has six edges, it was counted six times, and we divide by 6 to get five distinct tetrahedra. An alternate path to the same answer is that given a vertex, there are four possible triangles it is not on and therefore four tetrahedra including it. That would give 5 times 4 tetrahedra, but since each one has four vertices, we divide by 4 to get five distinct tetrahedra once again.

3. A tetrahedron surrounded by four more tetrahedra, one attached to each face

21.3 Coordinates and Cross Polytopes

Challenge The dual has eight vertices, each connected with an edge to six other vertices—all the others except the one it is "opposite." There are 24 edges total in this 4D "cross polytope."

Q1 In the cube, vertices connected by an edge, such as (1, 1, 1) and (1, 1, –1), differ in only one coordinate.

Q2 Every edge is parallel to one of the axes, so only one coordinate can change along an edge. Given the coordinates of two vertices, if they differ in exactly one of the coordinate positions, there is an edge connecting the two vertices; otherwise, there is no edge. This applies to hypercubes in any number of dimensions, as long as their edges are parallel to the axes.

Q3 All edges are of length 2. A typical edge goes from (1, –1, –1, 1) to (1, –1, 1, 1), differing in only the third coordinate. The $y_1 - y_2$ term is the only nonzero difference, and $\sqrt{(-2)^2} = 2$.

Q4 From (1, 1, 1, 1) to (–1, –1, –1, –1) is 4, which is twice the edge length.

Q5 The regular octahedron. It can be placed with all vertices on the coordinate axes.

Q6 There are two vertices on each of the n axes, making $2n$ vertices total. There are

$$\frac{2n(2n-2)}{2} = 2n(n-1)$$

edges.

Q7 In any number of dimensions, the edge length is $\sqrt{2}$.

21.4 The 120-Cell

Challenge Three such pentagons are used in this activity. See Models 1–5.

Q1

Type	Number in model	Number of 4D cells represented
1	1	2
2	12	24
3	20	40
4	12	24
5	30	30
Total	—	120

Explorations 21

A. Make two parallel 4D hypercube models, and connect the corresponding vertices.

B. You need six vertices, each connected to the other five. An octahedron with all vertices connected to a central ball is one method. The central ball corresponds to a crossing point, not a vertex. For a crossing-free model of seven balls (each directly connected to the other six), start with a triangular pyramid with $\mathbf{g_1}$ base and $\mathbf{y_1}$ slanting edges. Also build a triangular pyramid with $\mathbf{b_1}$ base and $\mathbf{y_1}$ slanting edges. Place them apex-to-apex, with a common three-fold axis, and combine them so that they share one ball as their common apex. By adding three additional $\mathbf{b_1}$s, three $\mathbf{b_2}$s, and three more $\mathbf{g_1}$s, every ball can directly connect to the others. This models a 6D simplex. For a 5D simplex, eliminate one ball and its connecting struts.

C. The vertices of the 4D cross polytope are (±1, 0, 0, 0), (0, ±1, 0, 0), (0, 0, ±1, 0), or (0, 0, 0, ±1). Take one from each group to have four vertices each a distance $\sqrt{2}$ from each other. There are 16 ways to choose a set of four since we pick one of two choices four times. Each choice gives one of the 16 regular tetrahedral cells.

D. The form is dual to the 120-cell.

E. The model is like a hexagonal prism ($\mathbf{b_3}$ hexagons as bases, $\mathbf{y_3}$ vertical edges) with two central vertical axes pushed against each other. Each connects to alternating base vertices. (There is also another symmetric projection, centered on a face, but it is hard to make with the Zome System.)

F. First count that there are 10 faces in the 4D simplex and 24 faces in the 4D hypercube. Then observe that $V + F = E + C$, where C is the number of 3D cells. This holds for any convex 4D polytope. More systematically, this can be written $N_0 - N_1 + N_2 - N_3 = 0$, where N_i is the number of i-dimensional components. The alternating sign pattern continues in the n-dimensional generalization.

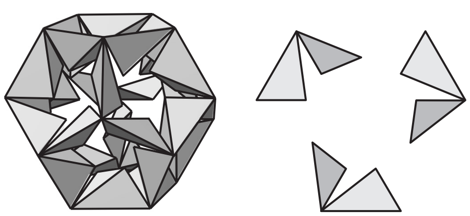
22.1 Stellated Polyhedra

Challenge In each case you see just the inside of an ordinary regular dodecahedron, which blocks your view of the rest of the structure.

Q1 The stella octangula is a stellation of the octahedron.

Q2 The 60 facets are grouped as 20 sets of three. Each set lies in a yellow plane. This is a stellation of the icosahedron whose vertices are at the apexes of the pyramids.

4., Q3



One face of a stellation of the icosahedron consists of six facets.

Q4 The interior vertices outline a rhombicosidodecahedron (3, 4, 5, 4).

22.2 More Stellations

Challenge Answers will vary.

Q1 To go from the first to the second stellation, 24 more rhombic pyramids are added.

Q2 To go from the second to the third stellation, another 24 pyramids are added. It can be seen as a compound of six identical irregular tetrahedra. One cannot build further and maintain the property that every exterior face is a continuation of an original rhombic dodecahedron face.

Q3 The planes of the faces of the stellation are those of the rhombic triacontahedron, and two faces meet at each edge.

8. The $\mathbf{b_2}$s are the points of a small stellated dodecahedron.

Q4 There are five cubes. This is another way of understanding the compound of five cubes studied in Unit 11.

Explorations 22

A. The face planes of the tetrahedron or cube, when extended, do not meet to form a new edge for the stellation.

B. All three structures meet the listed criteria. Each has 12 identical faces, in the planes of the dodecahedron's faces, and each has the same symmetry as the dodecahedron. Around a $\mathbf{b_1}$ dodecahedron, you can build a $\mathbf{b_4}$ small stellated dodecahedron, then a $\mathbf{b_3}$ great dodecahedron, and then a $\mathbf{b_6}$ great stellated dodecahedron.

C. Starting with a $\mathbf{g_1}$ cuboctahedron, $\mathbf{b_1}$ slanting edges make the pyramids over the triangular faces, and $\mathbf{g_1}$ slanting edges make the pyramids over the square faces. The result is the compound of a $\mathbf{2b_1}$ cube and a $\mathbf{2g_1}$ octahedron in dual position. The cuboctahedron is their intersection. Starting with the icosidodecahedron, the result is the compound of the icosahedron and dodecahedron. The icosidodecahedron is their intersection.

D. It is a stellation because it has the $\mathbf{r_1}$ rhombic triacontahedron at its core. It has the same number of faces as the $\mathbf{r_1}$ rhombic triacontahedron. Like the $\mathbf{r_1}$ rhombic triacontahedron, it has icosahedral symmetry. And all of its faces are identical. It is a polyhedron because two faces meet at each edge.

E. It is another stellation of the rhombic triacontahedron. Each face consists of two coplanar rhombi.

F. Each of the 5 tetrahedra contributes 4 of the 20 faces. Each lies in the yellow plane of an icosahedron's face.

G. One triangle from each of two different octahedra lies in each of the icosahedron's 20 face planes.

H. To start, make a $g_1 + g_2 + g_1$ square. In each corner, imagine the hypotenuse of the right triangle with g_1 legs. In a vertical plane, add another pair of g_1 legs to each imaginary hypotenuse. From these four right angles, insert four b_3s that meet at the center of the square. Turn this over, extend the right angles in the vertical plane into $g_1 + g_2 + g_1$ squares, and treat them like the first square. The approximation is off by about 2.3% because half the green edge is $2.568b_1$ and we use a b_3 of length $2.618b_1$.

23.1 Introduction to Fractals

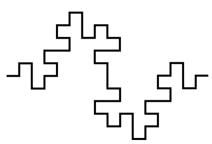
Q1 4^k struts are used.

Q2 $\frac{1}{3}$. Notice, for example, that the Stage 1 Zome model is three times as long as the Stage 0 model.

Q3 Take the logarithm (any base) of both sides to get $\log[3^x] = \log[4]$. Then $x \log[3] = \log[4]$, so

$$x = \frac{\log[4]}{\log[3]} \approx 1.26$$

3.



Stage 2

Q4 The number of struts scales by 8 as the length scales by 4, so the fractal dimension is

$$\frac{\log[8]}{\log[4]} = 1.5$$

23.2 Sierpiński in Three Dimensions

Q1 The number of pieces scales by 4 every time the length scales by 2, so it has dimension

$$\frac{\log[4]}{\log[2]} = 2$$

At the limit, this structure is two-dimensional!

23.3 More Three-Dimensional Fractals

Challenge At each level, the fractal consists of four copies of the previous level: a left-handed version, two right-handed versions, and one more left-handed version. The copies are connected by an extra line segment, which is indicated with a darker line in the figures.

1.–5. The complete path is R-F-L-U-R-N-L — U — R-U-L-F-R-D-L — F — R-U-L-F-R-D-L — D — N-D-F-R-N-U-F — R — N-D-F-R-N-U-F — U — L-U-R-N-L-D-R — N — L-U-R-N-L-D-R — D — L-F-R-D-L-N-R.

Q1 The $4 \times 4 \times 4$ cube can be doubled to a $4 \times 4 \times 8$, then $4 \times 8 \times 8$, then $8 \times 8 \times 8$, and so on.

Q2 There are as many edges as vertices, because they alternate, so a $p \times q \times r$ block contains pqr vertices.

Q3 The number of struts grows from 8 to 64, a factor of 8.

Q4 $\frac{\log[8]}{\log[2]} = 3$. It has a dimension of 3 corresponding to the fact that this curve fills a solid volume of space when the process is carried out indefinitely!

10. This is also a stack of prism + antiprism + prism, now all triangular.

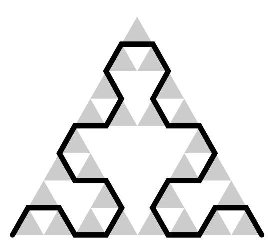
Explorations 23

A. At Stage 3, the central pentagram would have to have a side of length b_5. All the pentagrams are parallel, even though the branches they replace have five different directions.

B. The perimeter grows infinitely, as at each stage the new perimeter is $\frac{4}{3}$ of the previous perimeter. The area, on the other hand, is finite because the figure is bounded. The initial triangle has area $\frac{\sqrt{3}}{4}$. The first three bumps have combined area $\frac{\sqrt{3}}{12}$. From there on, each stage adds $\frac{4}{9}$ of the area added by the previous stage. At the limit, the total area of the bumps is $\frac{3\sqrt{3}}{20}$,and the total area enclosed by the snowflake is $\frac{2\sqrt{3}}{5}$.

C. The result is a curve that travels throughout an equilateral triangle but avoids all the holes in the Sierpiński gasket. It has dimension

$$\frac{\log[3]}{\log[2]} \approx 1.58$$



Stage 3 of Koch variation

D. It grows to fill much of a cube.

E. A cluster of 30 dodecahedra (or icosahedra) touching their neighbors along shared edges, surrounding a hollow icosidodecahedron

24.1 Proof of Euler's Theorem

Q1 $n = 0$: a ball alone. $n = 1$: any strut with a ball at each end. $n = 2$: three balls connected in a line or as two sides of a triangle.

Q2 $n = 3$: There are two possibilities: a snake arrangement, and a Y. $n = 4$: There are three possibilities: a snake, a starburst, and a Y with one arm having two segments.

Q3 No. A polygon has the same number of struts and balls, so it does not satisfy the definition.

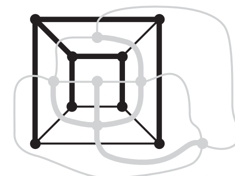
Q4–Q5 A tree never contains a loop. To see why, think of building any tree, starting with just a ball—the simplest tree. Then add a strut and another ball. This makes the next simplest type of tree. Keep going, by adding a strut and a ball at each step, for as many steps as you need. At each step we add one to the number of struts and the number of balls, so we always preserve the property that there is one more ball than strut. After each step, every strut has both ends inserted in some ball. In a tree, both ends of every strut always have a ball attached. There is never an opportunity to close a loop. This argument assumes that a complex tree can always be constructed by adding a strut and a ball to a simpler tree. This is so because every tree (except the $n = 0$ case) must have some ends. (The ends are called *leaves*—balls that touch only one strut.) If there were no leaves, every ball would touch at least two struts, which would imply by a counting argument that the number of struts is greater than or equal to the number of balls, contradicting the definition.

Q6 There are 12 balls, and we do not remove any. A tree has one fewer strut than ball, so our tree must have 11 struts. We must remove 19 of the original 30 struts. There are many choices as to which ones to pick.

Q7 There are 20 balls in a dodecahedron, and we do not remove any. A tree has one fewer strut than ball, so there are 19 struts in a tree chosen from a dodecahedron. We would remove 11 of the original 30 struts. There are many choices as to which ones to pick.

Q8 Yes. Starting with any icosahedron tree, this process always gives us a dodecahedron tree as well. As we saw in Question 7, the number of struts needed for a tree in a dodecahedron is 19. But having 19 struts is no guarantee that they form a tree. We also need to show they are connected and do not form loops. If you built your model correctly, this will be the case.

Q9



Sample Schlegel-based proof drawing for the cube and octahedron

24.2 Proof of Descartes' Theorem

Q1 $F = \frac{2}{3}E$ or $E = \frac{3}{2}F$. There are F triangles, which have a total of $3F$ edges. However, each edge is shared among two triangles and therefore was counted twice. So $E = \frac{3}{2}F$.

Q2 $V + F = \frac{3}{2}F + 2$, so $V = \frac{1}{2}F + 2$

Q3 There are $180F$ degrees of angle altogether, since the angles in each triangle sum to 180.

Q4 There would be $360V$ degrees of angle total, if it weren't for the deficit at each vertex. Taking all the deficits into account, we find the total of all the angles is $360V - D$.

Q5 $180F = 360V - D$

Q6 Solving the system

$$180F = 360V - D$$
$$V = \frac{1}{2}F + 2$$

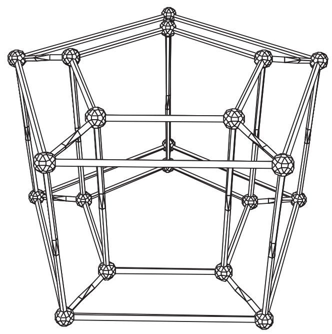
yields $D = 720$

Explorations 24

A. If the rhombic triacontahedron and icosidodecahedron are chosen, there are 31 edges in the first tree and 29 edges in the second.

B. $D = 0$ for a one-hole torus.

C.



Torus with five-fold symmetry

D. For a two-hole torus, $V + F = E - 2$ and $D = -720$.

Explorations 25

A. **a.** In the blue plane, there are 12 equatorial directions (4 blue, 4 red, and 4 yellow). You can make a 12-sided planar polygon in this plane, with one edge for each direction.

b. Green struts provide four more directions in this plane (45 degrees from the blues) allowing the construction of a planar 16-gon in the blue plane. However, there is an equilateral 18-gon constructible in the yellow plane. It is not exactly equiangular, but it is almost circular. Use six b_1s and 12 gb_1s, repeating in the pattern blue-green-green, always making the slightest turn possible, and taking care to remain in the plane.

B. There are only three equilateral pyramids. The base can be a 3-, 4-, or 5-gon. (With a 6-gon base, the pyramid would be flat, and with 7 or more sides, the slant edges would be too short to meet.) All are Zome-constructible: 3 and 4 are green, 5 is blue. Prisms and antiprisms on any regular *n*-gon can be equilateral in general, but few are Zome-constructible: the cube (which is also a 4-gon prism), a regular equilateral 8-gon prism, the octahedron (which is also a 3-gon antiprism), and the equilateral 5-gon antiprism.

C. Antiprisms with green struts:

regular octahedron

b_1 equilateral 3-gon bases with g_1 slanting edges

g_1 equilateral 3-gon bases with b_1 slanting edges (This is a cube deeply truncated at two opposite vertices, and it has isosceles right triangles for sides.)

g_1 equilateral 3-gon bases with y_1 slanting edges

b_3 5-gon bases with g_1 slanting edges (very flat)

b_1 5-gon bases with g_2 slanting edges (not Zome-constructible since two greens must go in the same red hole)

D. One with a three-fold axis of symmetry is b_3–g_2–b_3–g_2–b_3–g_2–. One with a two-fold axis is r_2–b_2–b_1–y_1–b_1–b_2–.

E. There are 16 possible triangles (counting similar triangles only once) that can be made with blue, red, and yellow: two right triangles (b_1–b_3–$2y_2$ and b_1–b_2–$2r_1$), eight isosceles triangles (b_2 base and anything but r_3 for a side), and six more in the blue plane (b_1–y_1–r_2, b_2–y_1–r_2, b_2–r_1–y_3, b_3–r_1–y_3, r_1–y_1–y_2, and r_1–r_2–y_2). (There are 39 more Zome triangles that contain green struts.)

F. If you take a zomeball and place 10 b_1s in the red plane and 10 y_1s in the yellow holes closest to this plane, you have the directions for the edges of the 20-gon. Using 20 r_3s to connect the 20-gon to a copy of itself makes for a stable wheel. For a 12-sided wheel, place six b_1s in the yellow plane and six r_1s in the red holes closest to this plane, and use those directions to make an almost-planar 12-gon. Use 12 y_3s to connect it to a parallel base. There are several analogous forms you can make near the blue equator, but they won't have equal dihedral angles; compare how they roll.

G. From inside to outside: $2b_1$ icosahedron, $g_1 + g_2$ octahedron, $2(g_1 + g_2)$ tetrahedron, $2b_3$ cube, $2b_2$ dodecahedron. (There are two variants, according to whether the icosahedron sits left-handed or right-handed in the octahedron.)

H. A triple-scale equilateral triangle is naturally dissected into nine smaller equilateral triangles. Take these in groups of three to form three equal trapezoids for each of the faces of the icosahedron. To demonstrate this icosahedron and the dodecahedron with faces divided into the five trapezoids described, you can trace your fingers along the respective models.

I. **a.** the cube

b. the y_2 rhombic enneacontahedron, consisting of 90 yellow rhombi (60 fat, 30 skinny)

J. **a.–c.** Build the solids in sizes b_2 and b_3, and connect them with y_1 or r_1 struts that would meet at the common center if extended.

d. Build in sizes y_2 and y_3 and connect with b_1 and y_1.

e. Build in r_2 and r_3 and connect with y_2 and r_2.

K. **a.** Scale model of the zomeball

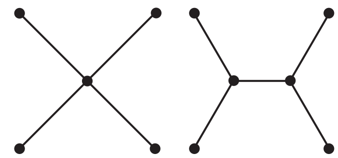
b. An arrangement of six skew decagons

c. With the rhombic dodecahedron, for example, you get a blue and green distorted rhombicuboctahedron and an interesting compound of four skew hexagons.

L. The ball looks like a rhombicuboctahedron, with 6 square holes, 8 triangular holes, and 12 rectangular holes. In the rhombicuboctahedron, opposite triangles are not parallel, so a twist is needed in the triangular struts. The square and rectangular struts do not need twists. See Peter Pearce's *Structure in Nature Is a Strategy for Design* for pictures and a description of one such construction system.

M. To the 31 directions of the zomeball, add 30 (5 times 12 divided by 2) green directions to get 61 directions, requiring 122 holes. The new ball would correspond to a polyhedron with 122 faces. The green holes lie between adjacent red and yellow holes. The simplest such zomeball is the zonish polyhedron described in Unit 18 as an icosidodecahedron expanded in ten yellow directions; it has skinny yellow rhombi in the blue directions and b–y–y–b–y–y hexagons in the green directions.

N. At most, three films meet at an edge in a bubble structure—never four or more—and they meet with dihedral angles of 120 degrees. The bubble films are like sheets of stretched rubber, which want to be as small as possible. The films self-adjust to minimize their total surface area. It can be shown (by advanced mathematical techniques) that minimal area films involve only 120-degree dihedral angles. So the only Platonic solids inside of which bubbles of the same shape form are those with three edges per vertex—the cube, dodecahedron, and tetrahedron. Suppose four films met at an edge, with dihedral angles of 90 degrees. A cross section would look like the figure.



Cross section of bubble films

The illustration on the right shows what happens if the film splits into 2 three-way joints (with 120-degree angles) rather than 1 four-way joint, again in cross section. Suppose that the edge of the outer square is of length 1 in each case. Then the four lengths in the illustration at left add to $2\sqrt{2}$, roughly 2.828, while the five lengths in the illustration at right add to $1 + \sqrt{3}$, roughly 2.732. The latter is smaller, so the bubbles choose it as a better structure, spontaneously splitting a four-way edge into 2 three-way edges with a new connecting surface.

O. This red and yellow nonconvex polyhedron has 120 red and yellow parallelograms and 60 red rhombi arranged like a giant flower ball.

P. The result is a b_3 icosidodecahedron with pentagrams in the pentagonal faces. Each equatorial decagon has a b_2 decagon within it, with edges extended to meet the b_3 decagon. Each brick has edges b_3, b_4, and b_5. Half the sphere makes a very sturdy dome.

Q. There are 60 squares (5 in each of 12 saucers) and 200 triangles (120 in antiprisms, 60 in saucers, and 20 on the three-fold axes).

R. The icosidodecahedron has 30 vertices and each connects to four neighbors. The vertices of a b_1 icosidodecahedron can be used to locate the centers of 30 spheres of radius $\frac{1}{2}b_1$, each of which contacts four neighbors. The distance from the center of the polyhedron to a vertex is b_2, so a central sphere in contact with the 30 small spheres has radius $b_2 - \frac{b_1}{2}$. The ratio of large sphere radius to small is then

$$\frac{\tau - \frac{1}{2}}{\frac{1}{2}} = 2\tau - 1 = \sqrt{5}$$

For other Sangaku problems, see "Japanese Temple Geometry," *Scientific American,* May 1998.

S. The Archimedean solids have identical vertices, so the vertices are all the same distance from the center. But they have more than one face type. The larger faces will have centers closer to the center of the sphere. The duals to the Archimedean solids, such as the rhombic dodecahedron and the rhombic triacontahedron, have identical faces. Their faces are all the same distance from the center, but they have two types of vertices at different distances from the center.

T. Cube: Divide one face of a $2b_1$ cube into eight congruent b_1–b_1–g_1 triangles that meet at the face center, and connect each triangle to the center of the cube with a b_1, a g_1, and a y_1. The orthoschemes for the other Platonic solids are defined similarly but cannot be constructed with the Zome System. In the tetrahedron there are 24 orthoschemes, in the octahedron there are 48 (the same number as the cube), and in the icosahedron and the dodecahedron there are 120.

U. If there were 2 five-fold axes, spinning one around the other, a fifth of a revolution would move the axis into the position of another five-fold axis, so there would have to be at least 6. Other than 1 and 6, it turns out that the only other possibility is to have infinitely many, as in a sphere, where every diameter is an *n*-fold axis for all *n*. Similarly, there can be 0, 1, or 4 three-fold axes (as in a brick, a triangular prism, or a cube) but not 2 or 3 three-fold axes.

V. Any plane that contains a two-fold or four-fold axis is a solution, so there are infinitely many, although most are not Zome-constructible. Constructible planes arise from slicing through the center with a plane perpendicular to any of the symmetry axes, except for the three-fold axes in the case of the tetrahedron. Slicing the cube or octahedron perpendicularly to the two-fold or four-fold axes gives its 9 mirror planes. But they can also be sliced perpendicularly to each of the 4 three-fold axes to obtain two halves with a hexagonal cross section. For the icosahedron and the dodecahedron, slicing perpendicularly to the two-fold axes gives the 15 mirror planes, but they can also be sliced in half perpendicularly to five-fold axes (revealing regular 10-gons) or three-fold axes (revealing regular 6-gons in the dodecahedron or irregular 12-gons in the icosahedron). For the tetrahedron, there are 6 mirror planes and 3 cuts perpendicular to the two-fold axes, revealing squares.

W. Paper disphenoids cannot be made from right triangles, because they end up flat. Nor can they be folded from obtuse triangles, because the two smaller angles sum to less than the largest angle and so cannot close at a vertex. A disphenoid with four isosceles faces can be made from two b_x struts and four y_x struts. There are no other Zome-constructible disphenoids except the regular tetrahedron.

X. **a.** The small stellated dodecahedron is 0-equivalent to the icosahedron and 1-equivalent to the great icosahedron.

b. Find the 8-gons in the rhombicuboctahedron's edges. Note that 6 squares (the blue ones) share edges only with squares, while 12 others (the green-and-blue ones) share edges with squares and triangles. Remove the triangles and 6 squares to get a (4, 8, 4, 8), or remove the 12 squares to get a (3, 8, 4, 8). (One can also find pairs of *3-equivalent* 4D objects.)

Y. The final form is like an r_2 rhombic triacontahedron, with two r_1s making a bridge over each rhombus's short diagonal, and the r_1s continued with r_2s to make three-fold points. It is dual to the $\left(3, \frac{5}{2}, 3, \frac{5}{2}\right)$ and is another stellation of the rhombic triacontahedron.

Z. The irregular decagon shapes are: b_2–r_2–y_2–y_2–r_2–b_2–r_2–y_2–y_2–r_2 and b_2–y_2–r_1–r_1–y_2–b_2–y_2–r_1–r_1–y_2. Tunnels of regular 10-gons pass completely through it in six directions. The large central tunnel is surrounded by ten smaller tunnels and then ten even smaller tunnels. After this, try three of its cousins: the 120-cell truncated to its edge midpoints, the truncated 600-cell, and the 600-cell truncated to its edge midpoints.

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Index of Polyhedra

Bolded entries indicate sections where building directions can be found.

Polyhedron (or Figure)	Materials Needed	Section
(3, 10, 3, 10) small icosihemidodecahedron	60 $\mathbf{b}_x$, 30 balls	**20.3**
(3, 10, 5, 10) small dodecicosidodecahedron	120 $\mathbf{b}_x$, 60 balls	**20.3**
(4, 10, 4, 10) small rhombidodecahedron	120 $\mathbf{b}_x$, 60 balls	**20.3**
(5, 10, 5, 10) small dodecahemidodecahedron	60 $\mathbf{b}_x$, 30 balls	**20.3**
120-cell	180 $\mathbf{b}_2$, 200 $\mathbf{y}_2$, 180 $\mathbf{r}_1$, 180 $\mathbf{r}_2$, 330 balls	**21.4,** 25E
16-hedron	30 $\mathbf{b}_1$, 12 $\mathbf{b}_2$, 20 balls	**5E**
16-hedron (space-filling)	18 $\mathbf{g}_x$, 12 $\mathbf{y}_x$, 16 balls	**16E**
31-zone zonohedron	360 $\mathbf{b}_1$, 240 $\mathbf{y}_1$, 120 $\mathbf{r}_1$, 480 balls	**14E**
600-cell	120 $\mathbf{b}_2$, 120 $\mathbf{y}_2$, 72 $\mathbf{r}_1$, 72 $\mathbf{r}_2$, 75 balls	**21E**
antiprisms	varies	**1.2,** 1.3, 1E, 4E, 5.1, 5.2, 6.1, 6.2, 9.3, 10.1, 12.2, 14E, 17E, 23.3, 25E
bilunabirotunda	26 $\mathbf{b}_x$, 14 balls	**16E**
compound of 3 cubes	48 $\mathbf{g}_1$, 24 $\mathbf{g}_2$, 12 $\mathbf{b}_3$, 54 balls	**22E**
compound of 5 cubes	60 $\mathbf{b}_1$, 120 $\mathbf{b}_2$, 30 $\mathbf{b}_3$, 80 balls	**11.2,** 11E, 20E, **22.2**
compound of 5 octahedra	180 $\mathbf{g}_1$, 240 $\mathbf{g}_2$, 302 balls	**11.3,** 22E
compound of 5 rhombic dodecahedra	240 $\mathbf{y}_2$, 120 $\mathbf{y}_1$, 170 balls	**11E**
compound of 5 tetrahedra	30 $\mathbf{b}_1$, 30 $\mathbf{g}_2$, 20 balls	**11.3,** 11E, 22E
compound of cube and octahedron	24 $\mathbf{b}_1$, 24 $\mathbf{g}_1$, 26 balls	**9.2,** 22E
compound of cuboctahedron and rhombic dodecahedron	48 $\mathbf{g}_1$, 72 $\mathbf{y}_1$, 60 balls	**12E**
compound of icosahedron and dodecahedron	60 $\mathbf{b}_1$, 60 $\mathbf{b}_2$, 62 balls	**9.2,** 22E, 24.1
compound of icosidodecahedron and rhombic triacontahedron	120 $\mathbf{b}_2$, 60 $\mathbf{r}_1$, 60 $\mathbf{r}_3$, 122 balls	**9E,** 12E
cube	12 $\mathbf{b}_x$, 8 balls	1.2, 2.2, 3.2, 3.3, 3.4, 3E, 4.1, 4E, 5.1, 5.2, 5.3, 6.1, 9.2, 10.1, 10.2, 11.1, 11.2, 11.3, 11E, 12.2, 12.3, 12E, 13.1, 13.3, 13E, 14.1, 14.3, 14E, 16.1, 16.2, 16.3, 16E, 17.1, 17.2, 19.1, 19.2, 19E, 21.1, 21E, 22E, 23.3, 23E, 25E
cuboctahedron	24 $\mathbf{g}_x$, 12 balls	**3.4,** 3E, **12.3,** 12E, 16.1, 16.2, 16E, 19.2, 22E, 24E, 25E
cuboctahedron with 4 zones	48 $\mathbf{g}_x$, 48 $\mathbf{y}_x$, 48 balls	**18E**

Polyhedron (or Figure)	Materials Needed	Section
icosahedron with 10 zones	180 y_3, 60 b_3, 132 balls	**18.2**
icosahedron, 2 stellations	#1: 30 b_3, 60 b_2, 60 g_2, 30 g_1, 80 balls #2 (1st stellation): #1 + 60 g_x, 32 balls	**22.1**
icosidodecahedron	60 b_x, 30 balls	3E, **12.1, 12.3,** 12E, 19.2, 20.3, 22E, 24E, 25E
icosidodecahedron with 6 zones	120 b_3, 120 r_3, 120 balls	**18.2**
icosidodecahedron with 10 zones	120 b_3, 182 y_3, 180 balls	**18E,** 25E
icosidodecahedron, stellated	120 b_3, 60 b_2, 62 balls	**22E**
kite-ohedron	varies	**9.3, 9E**
Koch curve	Stage 3: 64 b_1, 65 balls Stage 4: 256 b_1, 257 balls	**23.1,** 23E
Koch curve variation	Stage 3: 27 b_x, 26 balls Stage 4: 81 b_x, 80 balls	**23E**
octahedron, regular	12 g_x, 6 balls	**3.2,** 3.3, 3.4, 3E, 4.1, 4E, 5.1, 5.3, 6.1, 6.2, 9.2, 9E, 10.1, 11.3, 11E, 12.2, 12.3, 12E, 13.3, 13E, 14E, 16.1, 16.3, 16E, 19.2, 20E, 21.3, 21E, 22.1, 23.2, 24.2, 25E
octet truss	varies	**16.3, 18.3**
orthoscheme	3 b_1, 2 g_1, 2 y_1, 5 balls	**25E**
parallelepiped	12 struts (any color), 8 balls	**3E, 13.3,** 14.2, 14.3, 16E, 19E, 25E
pentagrammatic concave trapezohedron	10 b_1, 30 b_2, 22 balls	**20E**
pentagrammatic crossed antiprism	20 b_1, 40 b_2, 30 balls	**20.3,** 20E
prismatoid	varies	**17E**
prisms	varies	**1.2, 1E,** 3.1, 4E, 5.1, 5.2, 6.1, 6.2, 9E, 10.1, 12.2, 13.3, 14.3, 16.1, 16E, 17.1, 17.2, 17E, 18.3, 20.2, 21E, 23.3, 25E
pyramids	varies	**1.2, 1E, 3.1,** 4E, 5.1, 5.2, 6.1, 6.2, 9.3, 10.1, 12.2, 16.1, 16.2, 16.3, 17.2, 17E, 18.3, 21E, 22.1, 22.2, 22E, 23E, 25E
pyritohedron	36 b_x, 26 balls	**5.3**
rhombic dodecahedron 1	24 y_x, 14 balls	**11E,** 12E, **14.1,** 14.2, 14.3, 14E, 16.1, 17.2, 21.1, 22.2, 24E, 25E
rhombic dodecahedron 1, 2 stellations	#1: 48 y_x, 24 b_x, 26 balls #2: #1 + 72 y_x, 24 balls	**22.2**
rhombic dodecahedron 1, elongated	24 y_x, 4 b_x, 18 balls	**16.1**
rhombic dodecahedron 2	24 r_x, 14 balls	**14.1,** 14.2, 14.3, 14E, **17.1**

Polyhedron (or Figure)	Materials Needed	Section
truncated icosahedron	90 b_x, 60 balls	3E, 4E, 10E, **12.3,** 12E, 19.2, 19E, 25E
truncated icosidodecahedron	180 b_x, 120 balls	10E, **12.3,** 12E, 14.3, **18.2,** 25E
truncated octahedron	36 g_x, 24 balls	**3.4, 3E, 12.3,** 12E, 14.3, 16.1, 16E, 18.3, 25E
truncated rhombicuboctahedron	144 b_x and gb_x struts, 96 balls	16E
truncated tetrahedron	18 g_x, 12 balls	**3E, 12.3,** 16E, 19.2, 25E
zonohedron, polar	varies	**14.1,** 14.2, 14.3, 14E

Zome Struts