Geometrical Imagination

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Chapter I. Regular Objects

I. Regular Objects

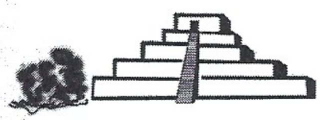
I.1 History and Imagination

In a telephone call to a radio talk show, the caller, an elderly man, poured forth his disappoint in life. He had been told in his youth that wisdom comes with old age. But wisdom did not come to him. He had been betrayed. No one told him that there is a path towards wisdom that combines imagination and reasoning.

Some people, associate geometrical ideas with philosophy, and even speak of "sacred geometry." It is interesting to hear their ideas; t

heir imagination had been strengthened by pondering patterns in buildings, symbols, legends, and geometrical ideas. Meanwhile geometry has been neglected in schools and colleges.

In this chapter there will be an example that involves a Swiss schoolteacher who went from ancient temples to four-dimensional geometry, in the 19th century. There will be an account of his interest in temples, followed by the foundational objects of two and three dimensions



In the ancient epoch of the Tigris – Euphrates Rivers, and in the structures of Central America there were temples of stepped pyramids. King Solomon sought something different for Jerusalem. He solicited the aid of the Phoenician King Hiram of Tyre, who was willing to help Solomon build his temple. King Hiram had a man skilled in architecture and construction, who was also named "Hiram". To avoid confusing the contractor with the King Hiram, the Hiram the builder has been called "Huramabi" or "Hiram-abiff." Here is a Bible translation from *Second Chronicles*.

> Solomon sent word to Hiram the king of Tyre: As you dealt with David my father and sent him cedar to build himself a house to dwell in, so deal with me. Behold, I am about to build a house for the name of the Lord my God and dedicate it to him for the burning of incense of sweet spices before him, and for the continual offering of the showbread, and for burnt offerings morning and evening, on the Sabbaths and the new moons and the appointed feasts of the Lord, our God, as ordained forever for Israel. The house which I am to build will be great, for our God is greater than all gods. But who is able to build him a him a house since heaven, even highest heaven, cannot contain him? Who am I to build a house for him, except as a place to burn incense before him? So now send me a man skilled to work in gold, silver, bronze, and iron, and in purple, crimson, and blue fabrics, trained also in engraving, to be with the skilled workers who are with me in Judah and Jerusalem, whom David my father provided. Send me also cedar, cypress, and algum timber from Lebanon, for I know that your servants know how to cut timber in Lebanon. And my servants will be with your servants, to prepare timber for me in abundance, for the house I am to build will be great and wonderful. I will give for your servants, the hewers who cut timber, twenty thousand cors of crushed wheat, twenty thousand cors of barley, twenty thousand baths of wine, and twenty thousand baths of oil.
>
> Then Hiram the king of Tyre answered in a letter which he sent to Solomon:

"Because the Lord loves his people he has made you king over them." Herum also said, "Blessed be the Lord God of Israel, who made heaven and earth, who given King David a wise son, endued with discretion and understanding, who built a temple for the Lord, and a royal palace for himself.

"Now I have sent a skilled man, endued with understanding, Huramabi the son of a woman of the daughters of Dan, and his father was a man of Tyre.

He is trained to work in gold, silver, bronze, iron, stone, and wood, and in purple, blue, and crimson fabrics and fine linen, and to do all sorts of engravings and execute any design that may be assigned him, with your craftsmen, the craftsmen of my lord, David your father. Now therefore the wheat and barley, oil and wine, of my lord has spoken, let him send to his servants; and we will cut whatever timber you need from Lebanon, and bring it to you in rafts by sea to Joppa, so that you may take it up to Jerusalem."

Then Solomon took a census of all the aliens who were in the land of Israel after the census of them which David his father had taken; and there were found a hundred and fifty-three thousand six hundred. Seventy thousands of them he assigned to bear burdens, eighty thousand to quarry in the hill country, and three thousand six hundred as overseers to make the people work.

Then Solomon began to build the house of the Lord in Jerusalem on Moriah, where the Lord had appeared to David his father, at the place David had appointed, on the threshing floor of Ornan the Jebusite. He began to build in the second month of the fourth year of his reign. These are Solomon's measurements for building the house of God: the length, in cubits of the old standard was sixty cubits, and the breadth twenty cubits. The vestibule in front of the of the nave of the house was twenty cubits long, equal to the width of the house; and its height was a hundred and twenty cubits. He overlaid it on the inside with pure gold. The nave he lined with cypress, and covered it with fine gold, and made palms and chains on it. He adorned the house with settings of precious stones. The gold was gold of paravain. So he lined the house with gold — its beams, its thresholds, its walls, and its doors; and he carved cherubim on the walls.

And he made the most holy place; its length, corresponding to the breadth of the house, was twenty cubits, and its breadth was twenty cubits; he overlaid it with six hundred talents of fine gold. The weight of the nails was one shekel to fifty shekels of gold. And he overlaid the upper chambers with gold.

In the most holy place he made two cherubim of wood and overlaid them with gold. The wings of the cherubim together extended twenty cubits: one wing if the one, of five cubits, touched the wall of the house, and its other wing, of five cubits, touched the wing of the other cherub; and of this cherub, one wing, of five cubits, touched the wall of the house, and the other wing, also of five cubits, was joined to the wing of the first cherub. The wings of these cherubim extended twenty cubits; the cherubim stood on their feet, facing the nave. And he made the veil of blue and purple and crimson fabrics and fine linen, and worked cherubim on it.

In front of the house he made two pillars thirty-five cubits high, with a capital of five cubits on the top of each. 1He made chains like a necklace and put them on be tops of the pillars; and he made a hundred pomegranates, and put them on the chains. He set up the pillars in front of the temple, one on the south, the other on the north; that on the south he called Jachin, and that on the north Boaz.

He made an altar of bronze, twenty cubits long, and twenty cubits wide, and ten cubits high. Then he made the molten sea; it was round, ten cubits from brim to brim, and five cubits high, and a line of thirty cubits measured its circumference. Under it were figures of gourds, for thirty cubits, compassing the sea round about; the gourds were in two rows, cast with it when it was cast. It stood upon twelve oxen, three facing north, three facing west, three facing south, and three facing east; the sea was set upon them, and all their hinder parts were inward. Its thickness was a handbreadth; and its brim was made like the brim of å cup, like the flower of a lily; it held over three thousand baths. He also made ten layers in which to wash, and set five on the south side, and five on the north side. In these they were to rinse off what was used for the burnt offering, and the sea was for the priests to wash in.

And he made ten golden lampstands as prescribed, and set them in the temple, five on the south side and five on the north. He also made ten tables, and placed them in the temple, five on the south side and five on the north. And he made a hundred basins of gold. He made the court of the priests, and the great court, and doors for the court, and overlaid their doors with bronze; and he set the sea at the southeast comer of the house.

Huram also made the pots, the shovels, and the basins. So Huram finished the work that he did for King Solomon on the house of God: the two pillars, the bowls, the two capitals on the top of the pillars; and the two networks to cover the two bowls of the capitals that were on the top of the pillars; and the four hundred pomegranates for the two networks, two rows of pomegranates for each network, to cover the two bowls of the capitals that were upon the pillars. He made the stands also, and the layers upon the stands, and the one sea, and the twelve oxen underneath it. The pots, the shovels, the forks, and all the equipment for these Hu'ramabi made of burnished bronze for King Solomon for the house of the Lord. I

In 1717, a Masonic lodge in London announced its existence; until that time Freemasonry was unknown to the general public. Some of their scripted initiations involve Hiram Abiff's assistants, who were not mentioned in biblical scripture.

The porch of the Temple had two pillars, called *Jachin* and *Boaz*; the names were from the family of King Solomon's ancestors. The two columns seem to have been symbolic rather than carrying structural loads.

Now, make a jump forwards about three thousand years to Ludwig Schläfli, a little known Swiss geometer. Schläfli was mentioned by the geometer H.S.M. Coxeter in a highly reapected geometry book *Regular Polytopes*, which he said took twenty years to write. Without this mention of Schläfli, the Swiss geometer would have been completely forgotten. Coxeter began as follows.

Ludwig Schläfli was born in Grasswyl, Switzerland, in 1814. In his youth he studied science and theology at Berne, but received no adequate instruction in mathematics

Schläfli, whose picture is at the right, vividly imagined the Jerusalem Temple, inside and out. Now, returning to Coxeter's account:

From 1837 to till 1847 he taught at a school in Thun, and learnt mathematics in his spare time, working quite alone until his famous compatriot Steiner introduced him to Jacobi and Drichlet. Then he was appointed a lecturer in mathematics at the University of Bern, where he remained for the rest of his long life

The "Steiner" mentioned here was the Swiss mathematician "Jacob" and not the better known Austrian philosopher "Rudolf."

While teaching school, Schläfli translated the work of Italian geometers into German, and German into Italian. Jacob Steiner appreciated these efforts, and it led to Steiner's associations with the mathematicians in Berne. Now let us return again to Coxeter's biographical comments on Schläfli.

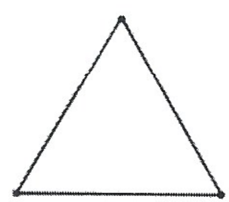
> His pioneering work was so little appreciated in his time that only two fragments of it were accepted for publication: one in France and one in England. However, his interest was by no means restricted to the geometry of higher spaces. He also did important research on quadratic forms, and in various branches of analysis, especially Bessel functions and hypergeometric functions; but he is chiefly famous for his discovery of the 27 lines and 36 "double sixes" on the general cubic surface. His portrait shows the high forehead and keen features of a great thinker. He was also an inspiring teacher. He used the Bernese dialect, and never managed to speak German properly. He died in 1895. ...
>
> The French and English abstracts of this work, which were published in 1855 and 1858, attracted no attention. This may have been because their dry-sounding titles tended to hide the geometrical treasures that they contain, or perhaps it was just because they were ahead of their time, like the art of van Gogh. Anyhow, it was nearly thirty years later that some of the same ideas were rediscovered by an American. The latter treatment (of Stringham) was far more elementary and perspicuous, being enlivened by photographs of models and by drawings. ... The result was that many people imagined Stringham to be the discoverer of the regular polytopes, as evidence that at last the time was ripe.

Polygons are lines that enclose a region of a plane. *Polyhedra* are plane surfaces that enclose a region of space. *Polytopes* of four dimensions are bounded by three dimensional polyhedra. It was Schläfli who first made an imaginary home in higher dimensions than we know by sensory experience.

I.2 Regular Polygons

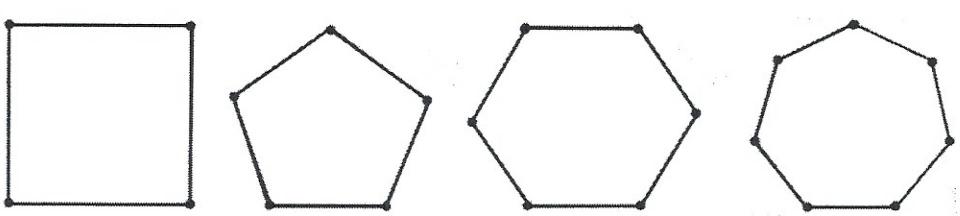
Polygons are figures in which points are connected by line segments.

Regular polygons have edges that are all of equal length and have equal angles at their junctions. The simplest is the equilateral triangle. The symbol {3} has other meanings, too; but It is the Schläfli symbol for equilateral triangles. (The name Schläfli would be pronounced like "shlayflee" in Switzerland.)



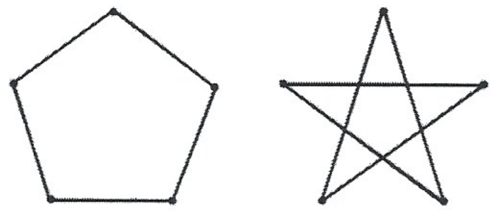
{3}

The Schläfli symbols for regular polygons are simply the number of edges written in braces.



{4} {5} {6} {7}

There are other polygons that have equal angles and equal line segments, but are not convex. *Convexity* means that when two any two interior points are connected the connecting line lies entirely within the figure.

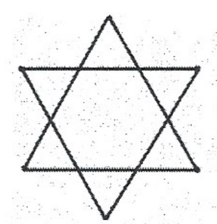


{5} $\{\frac{5}{2}\}$

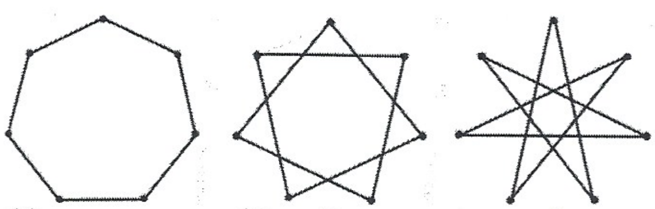
The edges of the regular pentagon, {5}, pass from one vertex to its neighbor. The edge of a *regular star pentagon* {5/2} skips past the closest corner and goes to the second. There is only one star pentagon, or *pentagram*. If you draw a {5/3} by skipping past two vertices you would merely be drawing the star {5/2} but going around in the opposite direction.

There are five-pointed stars in the flag of the United States; and there is a single star on flags of Communist countries.

The flag of Israel has a figure with six vertices. If we start at a vertex and pass by a neighboring point, going to the second one, then the figure breaks apart into two separate, isosceles triangles, each of them a {3}. Such an object is said to be a *compound* of . This compound is sometimes called "Solomon's seal" or the "Star of David."

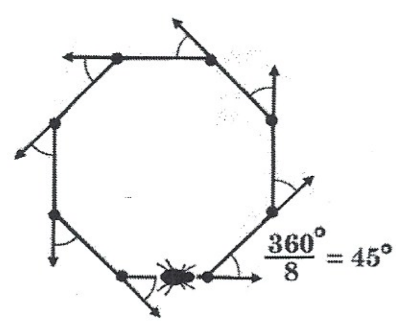


When there are 7 points the polygons do not fall apart into compounds; seven is a prime number, so there is a polygon and two legitimate star polygons. There are no compounds.



{7} $\frac{7}{2}$ $\frac{7}{3}$

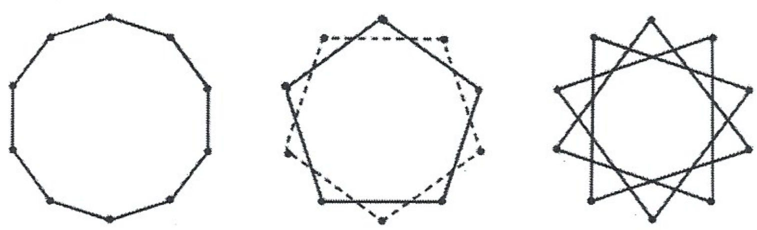
Now, consider a spider walking around the regular octagon. It is starting at the bottom. It travels in the direction of the arrows, but makes turns when it comes to a vertex. Returning to its original position, it must have turned 360°. Since there are eight turns in the diagram, each of the turns is 45°.



To obtain the interior angle we need merely to subtract 45° from the 180° angle along the inside direction of the arrow. This gives 180°– 45° = 135°.

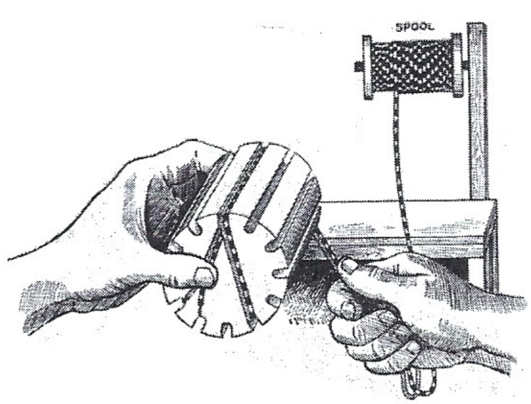
Schläfli symbol	Interior Angle (degrees)	Interior Angle (radians)
{3}	60°	$\pi/3$
{4}	90°	$\pi/4$
{5}	108°	$3\pi/5$
{6}	120°	$2\pi/3$
{7}	128 $^{4}/_{7}$°	$5\pi/7$
{8}	135°	$3\pi/4$
{9}	140°	$7\pi/9$
{10}	144°	$4\pi/5$
{11}	147 $^{3}/_{11}$°	$9\pi/11$
{12}	150°	$5\pi/6$
{n}	$180[(n-2)/n]$	$\pi[(n-2)/n]$

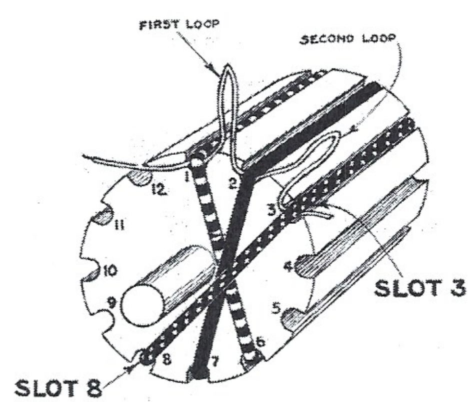
The table above gives the interior angle of a regular polygon, {n}. Similar computations give the angles measurements of star polygons.



{10} compound $\{\frac{10}{3}\}$

Regular polygons and regular star polygons are the simplest objects that can be named with the Schläfli symbol. Here are drawings from *Adel's New Electric Library*, 1929. There is an presence of star polygons in electric motor winding,

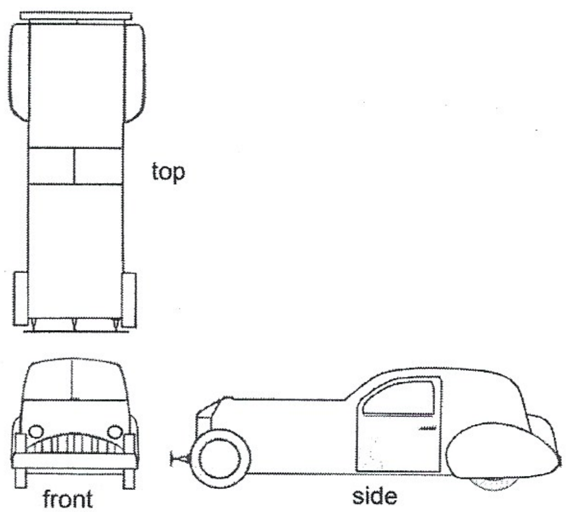




I.3 Descriptive Geometry

In *descriptive geometry* a three-dimensional object is shown by a front, top, and side view. The three views are intended that the reader imagines a three dimensional object.

In some countries the placement of the three drawings – top, front, and side – have a different arrangement; and sometimes extra views are needed.



After 1970 most draftsmen lost their jobs working for engineers and architects. Automatic machine shop work can be done without descriptive drawings, using databases instead. Nevertheless, for short run work drawings are still useful.

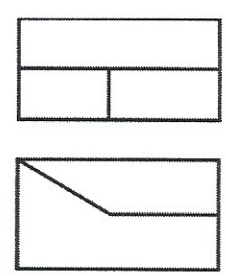
A described object is projected on three mutually perpendicular planes. The *top view* is carried onto a plane parallel to the ground, above the car. Points on the car are carried to the picture plane by parallel lines.

The side view is also called a “profile”.

Here is a sample problem. If the object is simple, you may be able to draw a third view (top, front, or side) knowing the other two views.

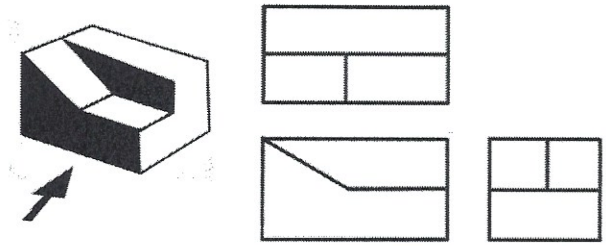
Sample Problem

The top and front view of an object are given.
Sketch the side view.

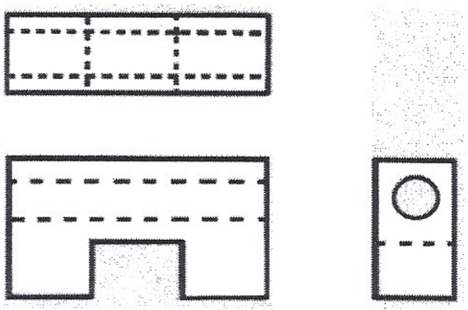


The front view shows a sloping cut-a-way block. The top view is above the front view. We must imagine the object from these two views. Draw the side view and read (that means sketch) the object. The arrow is inrended to show the direction of the front.

Solution.



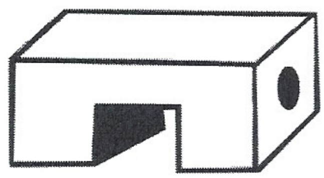
Sample Problem
Read the descriptive object.



The dashed lines represent hidden edges. The front view shows a tunnel through the object. This tunnel appears in the top view as dashed lines; in the side view the tunnel roof appears as a horzontal, dashed line.

The circular hole, seen in the side view is represented by horizontal dashed lines in the top view and the front view. Once this is imagined, we can *read* the object by sketching it in perspective.

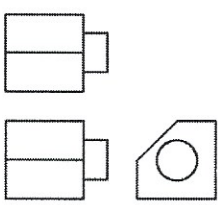
Solution.



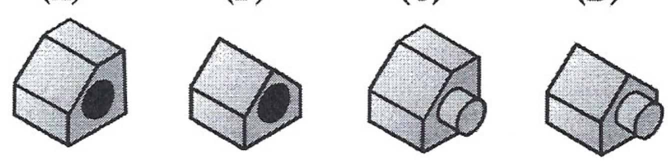
Problem Set I

In problems 1 and 2, chose the letter that is properly read from the descriptive diagram at the left.

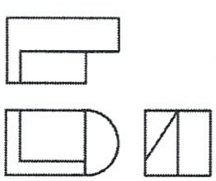
1.



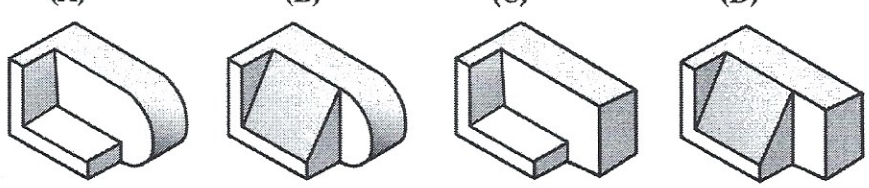
(A) (B) (C) (D)



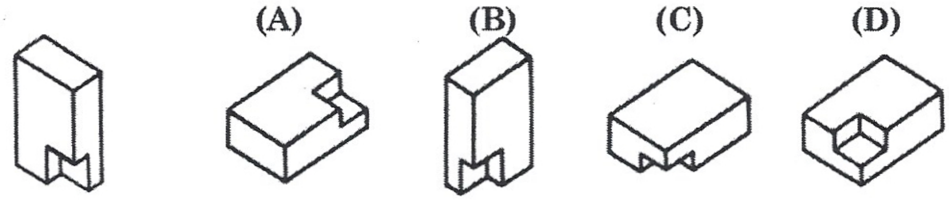
2.



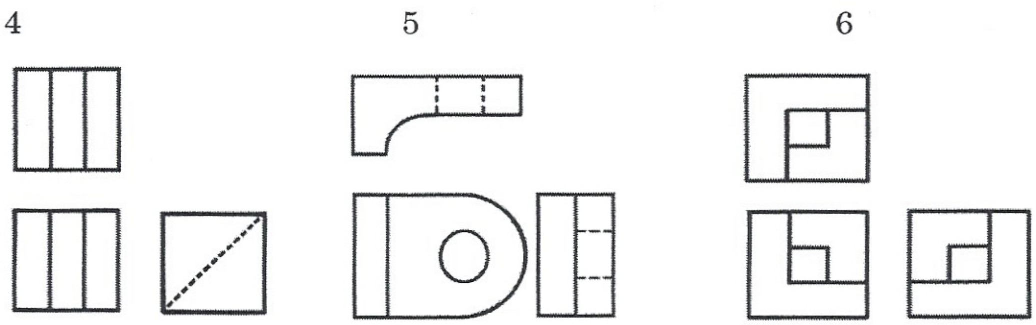
(A) (B) (C) (D)



3. An object is shown at the left. Choose the letter of the same object.



Read the descriptive diagrams in problems 4, 5, & 6.



I.4 Skew Lines

Two flags have been put up to mark the place for utility lines. A rabbit has dug near its hole, making one pole lean. The flag poles are line segments, but we can imagine extending them indefinitely, in a full line.

Two lines are said to be *skew* if they are not parallel and have no plane in common. A line cannot be skew by itself. Being *skew* is a relation involving two or more lines.

Now, imagine a plane passing through the upright flagpole. As the plane rotates around the upright pole it will be punctured in only one place by the leaning line.

1.5 Polyhedra and Tessellations

The Schläfli symbols that have two numbers are either regular three-dimensional objects, or infinitely spread tiles. The five *regular polyhedra* are also called *Platonic solids*. Archeologists found regular solids of stone were around the world.

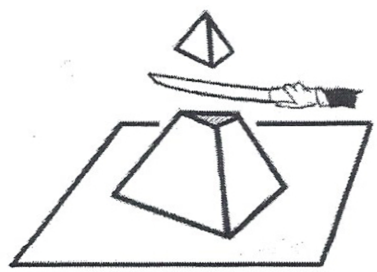
Here is list of regular polyhedral and regular tessellations that will be explained.

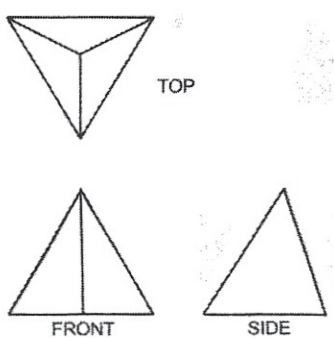
{3, 3} Tetrahedron	{4,3} cube
{3, 4} Octahedron	{4, 4} square tessellation
{3, 5} icosahedron	{5, 3} dodecahedron
{3, 6} triangular tessellation	{5, 3} hexagonal tesellation

I.6 Tetrahedron {3, 3}

The simplest polyhedron is the tetrahedron. Three points determine a plane and a fourth point off of that plane

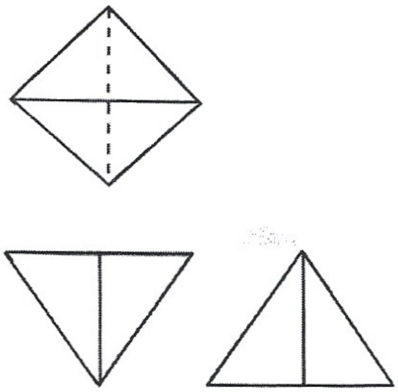
A *regular tetrahedron* has {3, 3} as its Schläfli symbol. The first "3" means that the faces are equilateral triangles; the second "3" represnts a *vertex figure.* When a slice is made beneath a peak, a triangle shows; that is a vertex figure. It gives the second "3" of the symbol {3, 3}.





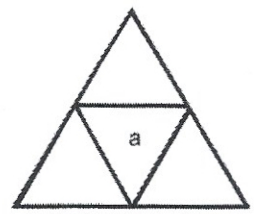
The TOP view of a regular tetrahedron, {3, 3} shows the true length of the edges on the triangular base. The edges that attach at the peak are foreshortened because they fall away from the top. The TOP view shows all four triangular faces.

The fourth triangular face in the FRONT view is lying flat and appears as the bottom line



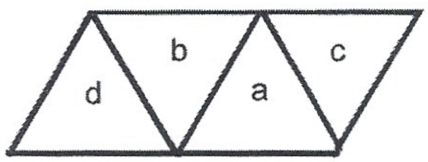
There are other ways to describe a regular tetrahedron. The dashed line is at the bottom; the tetrahedron is balanced on it, as if it were floating in water. The top and the bottom edges are skew. All six edges, and all four faces of the tetrahedron show in the top view

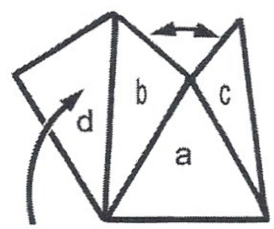
The front and side view have edges that are hidden behind other edges.



hen making a cardboard tetrahedron, with "a" as its base, the other three faces can be folded back

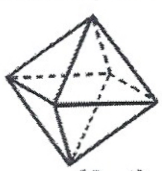
Gluing is somewhat easier when starting with a strip of four triangles. Fold triangles b and c, fasten them at the top. Triangle d folds back and then can be fastened in the remaining gap. The gluing tabs are not shown on the diagrams.





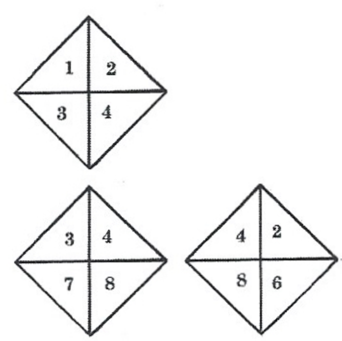
I.7 Octahedron {3, 4}

A regular octahedron has eight triangular faces, and six corners. The "3 in Schläfli symbol {3, 4} represents the triangle faces; the "4" tells that there are four triangles at a corner.

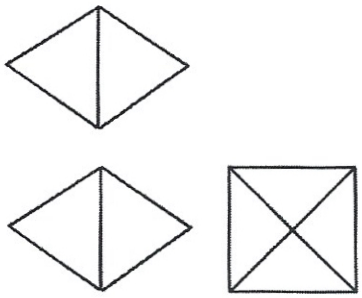


{3, 4}

Half a tetrahedron resembles a pyramid. In a corner-down position, the three views of its descriptive representation are all alike.



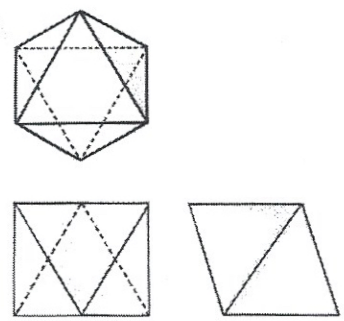
The corner down descriptive view of an octahedron shows only four of the eight triangles in each of the projections. Faces that have the same numbers are the same triangle.

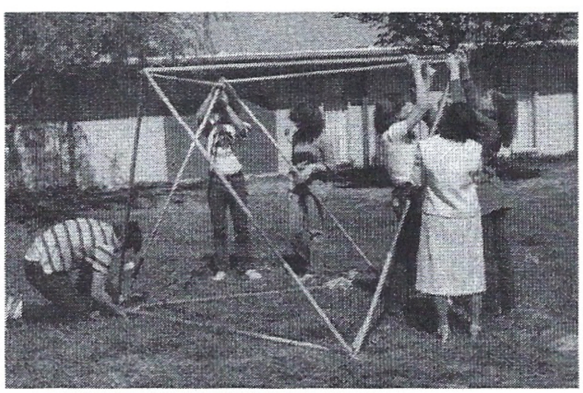


At the left is a description of an octahedron balancing on a single edge, as if it were floating.

The accompanying diagram of four triangles shows how to make half an octahedron from cardboard. When attached, as shown by the arrow, it resembles a pyramid, with an open square at the bottom. Two of these can be joined to make an octahedron.

When the octahedron is standing on a triangular face, the we see a hexagonal arrangement of edges that are not foreshortened. The six outer edges are shortened as they go from the top to the bottom triangle in dashed lines.



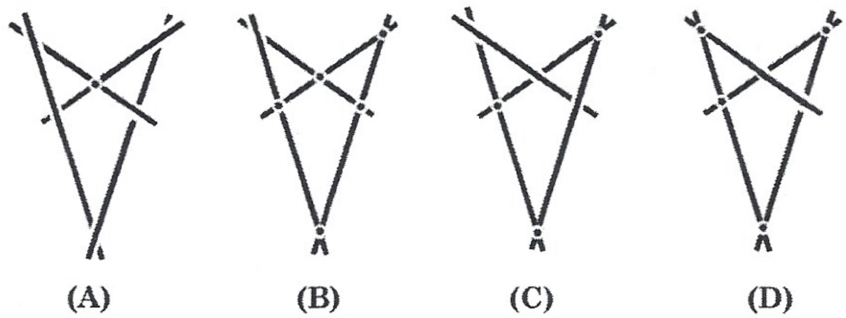


The photograph shows high school students assembling a large octahedron, with a triangular with its face on the ground. From the inside, a person has the experience of bodily movement. The pattern of six pairs of parallel edges show clearly.

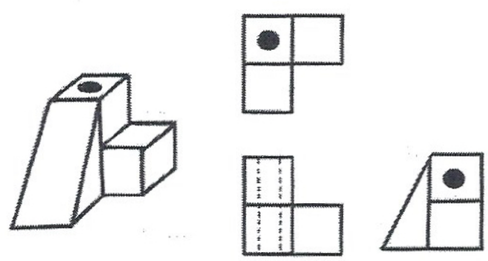
Problems II

1. Choose an edge of a tetrahedron. How many edges are skew to it?

2. Which configuration of lines *cannot* be brought into three dimensional space?

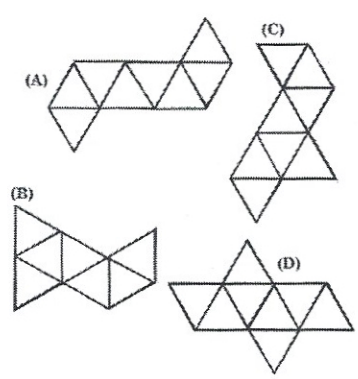


3. The object at the left has been drawn; but one of the descriptive views has an error. Find the error and correct it.

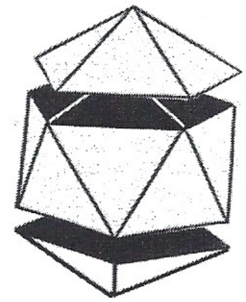


4. How many distinct pairs of skew lines can be found in a regular octahedron?

5. Which pattern of triangles cannot be folded to make an octahedron?

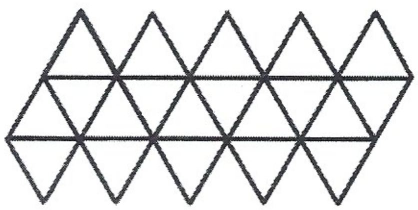


I.8 Regular Icosahedron {3, 5}

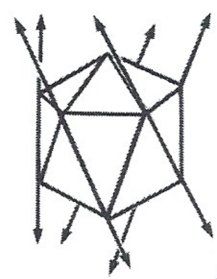


The regular icosahedron has twenty regular triangles, meeting five at a corner.

The easy way to imagine an icosahedron is to remove a cap with a vertex on top, then another ring of five triangles underneath. There remains drum-like ring of triangles in the middle that is composed of ten triangles alternately pointing up and down.



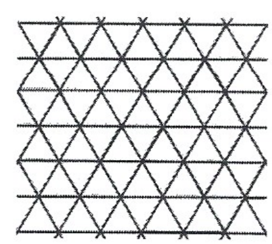
The ten flat triangles at the left can be folded and attached to make an icosahedron. The central belt of ten triangles produces the drum-like ring. Five triangles at the top and bottom make the two caps at the top and bottom.



An icosahedron can have five mutually skew edges. Pick two vertices that are opposite each other, one pointing up and the other down. On the drum-like ring between them, there two ways of locating five mutually skew lines. One set of skew lines turns clockwise and the other counter-clockwise.

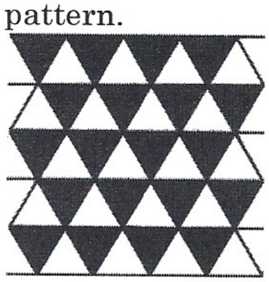
The remaining five edges form a second set of five skew lines that suggest a screw turn in the opposite direction. This can be built quickly using blue lines of the Zometool, an interesting way to make geometric models. .

I.9 Tiling {3, 6}



At the right are equilateral triangles are placed around a point. They make a tiling, or *tessellation*. Since $6 \times 60° = 360°$, the tiles lie flat at every corner.

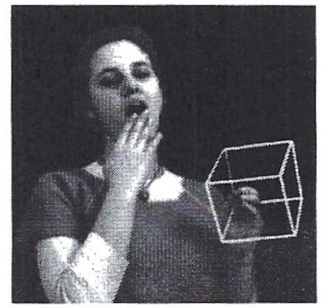
The pictures below has alternate black and white triangles. Four triangles meet to make a {3, 4} octahedron; six triangles {3, 6} male a tile pattern.



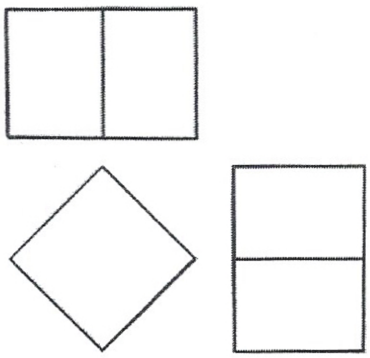
Both {3, 4} and {3,6} are related to the number two because they can be colored with two colors, with no edge having the same color on both sides.

I.10 The Cube {4, 3}

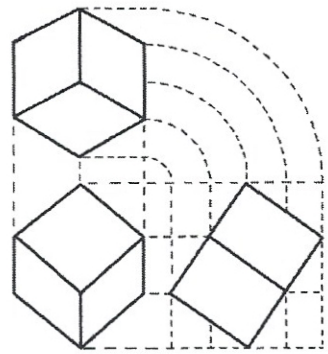
Until now all geometrical forms had triangular faces. A cube has six square faces, eight vertices, and twelve edges. The Schläfli symbol for a cube is {4, 3}. The "4" means there is a square (a polygon of four edges); "3" means that the square faces meet three at a corner. Some people find cubes boring.



(i)



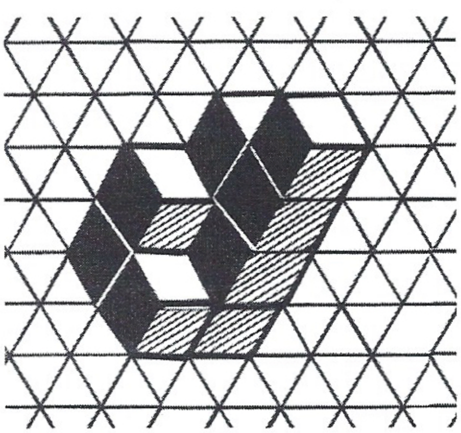
(ii)



Description (i) is a cube balanced on an edge. In (ii) there is a point with its corner down. The dashed lines in (ii) show how a third view can be drawn by using the distances from the other two views

There is also a way of drawing stacked blocks using the triangular grid. The stack that is shown below depicts ten blocks. It is intended that blocks are supported by other blocks underneath, even if they are not shown

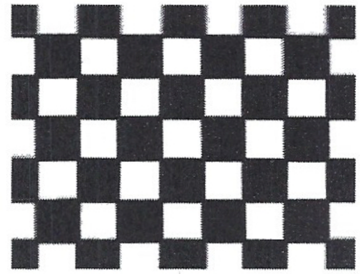
There are three blocks in the row on the left, and 3 + 4 = 7 on the right. Altogether there are ten blocks.



A picture of this kind, that is made on a triangular grid, is called *isometric.*

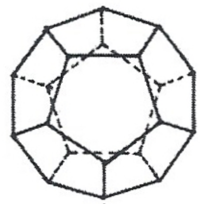
Imagine cubic blocks that are all the same size. Now, notice that a block, the uppermost one for example, depicts a cube with three mutually perpendicular edges; they are of equal length. This supports the use if the word *isometric.*

I.11 Tiling {4, 4}

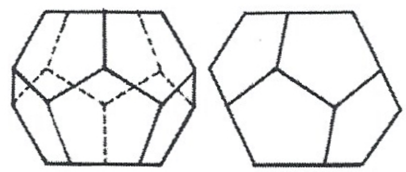


The Schläfli symbol tells is that these tiles have four equal sides and that they meet four at a corner. This checkerboard tiling here has two different colors on each edge of a tile.

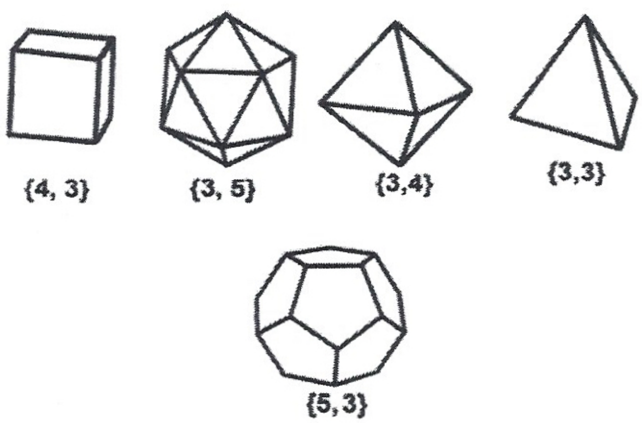
1.12 Dodecahedron {5, 3}



The regular dodecahedron consists of twelve pentagons. The picture shows a descriptive representative of a regular dodecahedron standing on one pentagonal face. They can be assembled quickly with a set of Zometools



1.13 History: Polyhedra



The five regular polyhedra, {3, 3}, {3, 4}, {3, 5}, {4, 3}, and {5, 3}, have been called the "Platonic solids", but they have been found carved out of stone throughout the world.

A peculiar aspect of the Platonic solids is that they have been given a correspondence with states of matter, which were solid, liquid, gas, and heat. Solid objects were said to be in correspondence with cubes, {4, 3}. The liquid state corresponded with the icosahedron, {3, 5}, presumably because it is the most round of the five; it can roll away easily. The gaseous state corresponded to the octahedron, {3, 4}.

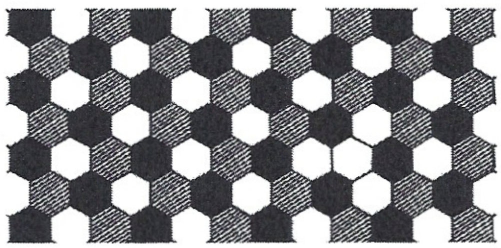
We would not regard heat as a state of matter today, but rather a state of energy; it has the sharpest angles.

The dodecahedron was not regarded as representing a state of matter, but it was a *quintessence*. The quintessence was in the havens, but it also permeated through earthly materials. It was a human connection with cosmic spirits. The quintessence disappeared after the Middle Ages, but the regular dodecahedron, {5, 3} remains with us. It is the only place that a "5" appears among the regular polygons or regular tessellations.

I.14 Tiling {6, 3}

There can be no regular polygon with hexagonal faces. Hexagons lie flat because $3 \times 120° = 360°$.

Three colors are needed if we want to avoid having the same colors next to each other .



This is the last of the Schläfli symbols of two digits.