The Divine Proportion in the Twentieth Century

David Booth, editor

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Biographical Notes
Hermann von Baravalle

Hermann von Baravalle (1898-1973) came from an aristocratic family that had close connections with the royal family the Austro-Hungarian Empire. He was swept into World War I, at the age of seventeen. While in the army he learned of anthroposophy, a philosophy developed by Rudolf Steiner, and made a trip to meet Steiner while he was on leave.

Steiner had been closely associated with the theosophical movement but had also written philosophical treatises that bear no trace of the concern with hidden masters that occupied the theosophical movement. In 1910? he had separated from the theosophical movement to form the anthroposophical movement. The root cause of this split was probably Steiner's growing interest in Christianity in contrast to the orientalism that had surrounded theosophy.

When the war ended the old Austria was gone. Von Baravalle was certainly in a psition to take his life in a new direction. He had begun studying mathematics at the University of Vienna. Meanwhile, Emile Molt, who directed the Waldorf Astoria cigarette company in Stuttgart, Germany, wanted to start a company school that would represent a new educational impulse for post-war reconstruction. He asked Rudolf Steiner to help in the project. Steiner suggested to von Baravalle that he become a mathematics teacher.

Von Baravalle returned to Vienna, wrote up some of his ideas on mathematics and physics education, obtained his Ph.d. in education, and joined the faculty of the first Waldorf School within a year of its opening in 1919. As the Second World War drew closer, German education became politicized and the Waldorf Schools that had spread from Stuttgart were closed. Dr. von Baravalle arrived in America in 1937, where he taught mathematics education and assisted in the formation of American Waldorf Schools.

Several articles written by von Baravalle appeared in the *Mathematics Teacher* during the 1940's. During these years he had cause to hope that the ideas developed in European Waldorf Schools might influence the teaching profession in

America. He viewed the new math movement that came around 1960 with deep skepticism, however.

The *Mathematics Teacher* has kindly given permission for the reprinting of this article on the golden section, from 1948.

The Geometry of the Pentagon and the Golden Section

Hermann von Baravalle

The Geometry of the Pentagon has become almost a foster-child besides the other chapters of geometry, as for instance the geometry of the triangles or of the quadrilaterals. Considering terminologies, we find the whole field of trigonometry deriving its name from the geometry of triangles and the "quadrature of areas" (quadratum = square) from the regular representative of the quadrilaterals, all units for measuring areas also being squares.

The characteristic elements of the geometry of the pentagon are neither related to the trigonometric reproduction of forms nor to measuring areas. The regular pentagon, however, and especially the regular stellar pentagram, are used today in the flags and emblems of the mightiest nations and had a similar use two and a half thousand years ago when the pentagram was the emblem of the Pythagorean School. It is the particular appeal of the pentagon to the sense of beauty, and the unique variety of mathematical relationships connected with it which are the characteristics of the geometry of the pentagon. Therefore, this geometry is particularly fit to stimulate mathematical interest and investigations. Outstanding among the mathematical facts connected with the pentagon are the manifold implications of the irrational ratio of the Golden Section.

The first figure shows a regular pentagon, and inscribed in it the pentagram formed by its diagonals. The central area of the pentagram forms again a regular pentagon in reverse position. In this pentagon another pentagram has been inscribed. The total diagram of Figure 1 contains three horizontal lines, among them the base of the pentagon. Due to symmetry there is a group of three parallel lines coordinated in the same way to every one of the five sides of the pentagon. These parallel lines form between them two types of rhombi, smaller and larger ones.

One of the smaller and one of the larger rhombi is marked in Figure 2 and Figure 3. A diagonal divides a rhombus into

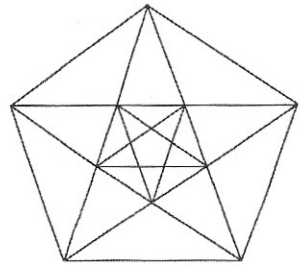


Figure 1

two congruent isosceles triangles. By folding and bending over the marked rhombus in Figure 2 along its horizontal diagonal we shall always reach exactly the opposite vertex of the central area. Cutting a pentagram out of paper, then bending over its outer parts and holding the paper before a light will make the inner pentagram appear in the central area. Folding the marked rhombus of Figure 3 in the same way, along its horizontal diagonal, will bring into coincidence two of the angles which divide the interior angles of a regular pentagon. Consequently, a pentagram trisects the angles of a circumscribed pentagon. If one of the partial angles is denoted by ϕ the angles of the large rhombus of Figure 3 are 2ϕ; 3ϕ; 2ϕ; 3ϕ; and those of the small rhombus in Figure 2: ϕ; 4ϕ; 4ϕ; ϕ.

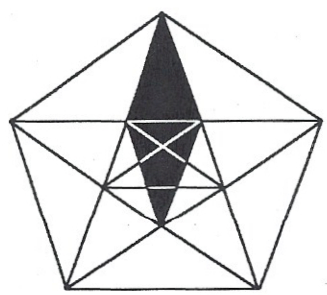


Figure 2
Smaller Rhombus

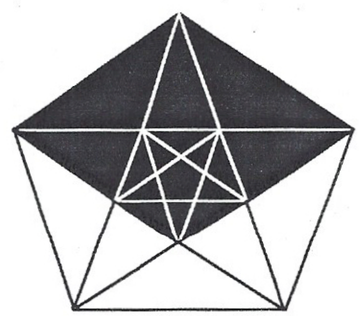


Figure 3
Larger Rhombus

The sum of the angles of either of the two rhombi being 10ϕ, 360°/10 = 36°. All angles which come in the diagrams of the Figures 1-3 are of the sizes: 36°; 72°;108° 144°; 180°; 216°; 252°; 288°; 324°; and 360°, forming an arithmetic progression with a difference of 36°.

The line segments in Figure 1, including both the partial segments between points of intersection and also their sums, are of six different sizes. The largest are the diagonals of the large pentagon. Regarding them as of size No. 1 and then continuing with numbering until we come to size number 6 with the sides of the inmost pentagram, the various sizes appear in the following quantities:

Line segments of size No. 1 come in the diagram	5 times:
Line segments of size No. 2 come in the diagram	15 times:
Line segments of size No. 3 come in the diagram	15 times:
Line segments of size No. 4 come in the diagram	15 times:
Line segments of size No. 5 come in the diagram	10 times:
Line segments of size No. 6 come in the diagram	5 times:
Total number of segments	65

In Figure 4, three isosceles triangles which are contained in the pentagram are marked by shading. The sides of the largest one are of the sizes No. 1; No. 1; No. 2. The sides of the middle sized triangle are No. 2; No. 2; No 3 and those of the smallest triangle are No. 3; No. 3; No. 4. In the complete diagram of Figure 1 further triangles of the same form are contained which are smaller and have sides of the sizes 5 and 6. The similarity of all these triangles establishes the following equations of the ratios of the line segments:

$$\frac{\text{segm1}}{\text{segm2}} = \frac{\text{segm2}}{\text{segm3}} = \frac{\text{segm3}}{\text{segm4}} = \frac{\text{segm4}}{\text{segm5}} = \frac{\text{segm5}}{\text{segm6}}$$

Therefore, the six sizes of line segments are members of a geometrical progression. Whereas the angles in the pentagon diagram make up an arithmetical progression the line segments form a geometric progression. Denoting the ratio of this geometric progression by "x" and writing "a" for the line segment of size 1, we have:

segment of size No. 1 = a
segment of size No. 2 = ax
segment of size No. 3 = ax^2
segment of size No. 4 = ax^3
. . . .
segment of size No. n = ax^n

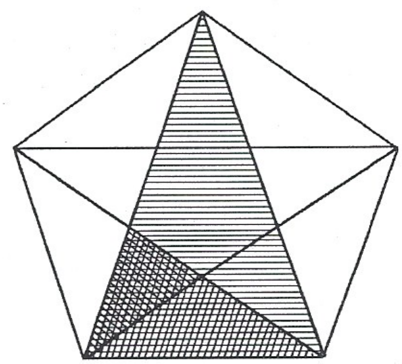


Figure 4 Similar triangles in a pentagon

The value of x can be found through the fact that one line segment of size 3 and one of size 2 make up a pentagram side of size 1. Therefore $ax^2 + ax + a$ or $x^2 + x = 1$. Solving the equation for x we get $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1}$. The positive root is

$$-\frac{1}{2}+\frac{\sqrt{5}}{2}=\frac{\sqrt{5}-1}{2}=0.61803398875\cdots$$

which is the number of the Golden Section G.

The expression $G = -\frac{1}{2} + \sqrt{\frac{1}{4} + 1}$ suggests the construction of a right triangle with the legs of one-half and of one unit. From its hypotenuse $\sqrt{\frac{1}{4} + 1}$ we subtract one-half unit and obtain the length of G units.
Starting the construction with any given line segment, one obtains the original length multiplied by the factor G. All the 65 line segments of the diagram in Figure 1 can thus be obtained from the large pentagram side by repeated application of the described construction.

Other lengths connected with the pentagram, for instance, the relative altitudes of its vertices can also be expressed through G. This can be done by applying the theory of the complex-number plane. The geometry of a regular n sided polygon reappears in the n^{th} roots of unity. For the pentagon, we use a 5^{th} root of unity that corresponds to the equation $x^5 - 1 = 0$. One of the roots being 1, we get through synthetic division:

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & \end{array}$$

and obtain the quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$. Applying to it the method of reciprocal equations, we first divide by x^2 and get:

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0$$

Then we regroup:

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0$$

Substituting y for $x + (1/x)$ and therefore y^2 for $x^2 + 2 + (1/x^2)$ or $y^2 - 2$ for $x^2 + (1/x^2)$ the equation takes on the form $y^2 + y = 1$ which is again the characteristic equation which has G as its root. The two roots $y = \dfrac{-1 \pm \sqrt{5}}{2}$ can be expressed using G as $\dfrac{-1 + \sqrt{5}}{2} = G$ and $\dfrac{-1 - \sqrt{5}}{2} = -(G + 1)$. The values for x are obtained by solving the equations:

$$x + \frac{1}{x} = \frac{-1 + \sqrt{5}}{2} = G \text{ and } x + \frac{1}{x} = \frac{-1 - \sqrt{5}}{2} = -(G + 1).$$

By multiplying the first equation with x we get: $x^2 - Gx = -1$. Its roots are

$$x = \frac{G}{2} \pm \sqrt{\frac{G^2}{4} - 1}$$

By multiplying the second equation with x we get:
$x^2 + (1 + G)x = -1$ and the roots are

$$x = -\frac{1+G}{2} \pm \sqrt{\frac{(1+G)^2}{4} - 1}$$

As $G < 1$ both $G^2/4$ and $(1 + G^2)/4$ are smaller than 1 and consequently all the four roots are complex. The five roots of the original equation $x^5 - 1 = 0$ are:

$$r_1 = 1 \qquad r_2 = \frac{G}{2} + \sqrt{1 - \frac{G^2}{4}} \cdot i$$

$$r_3 = -\frac{1+G}{2} + \sqrt{1 - \frac{(1+G)^2}{4}} \cdot i \qquad r_4 = -\frac{1+G}{2} - \sqrt{1 - \frac{(1+G)^2}{4}} \cdot i$$

$$r_5 = \frac{G}{2} - \sqrt{1 - \frac{G^2}{4}} \cdot i$$

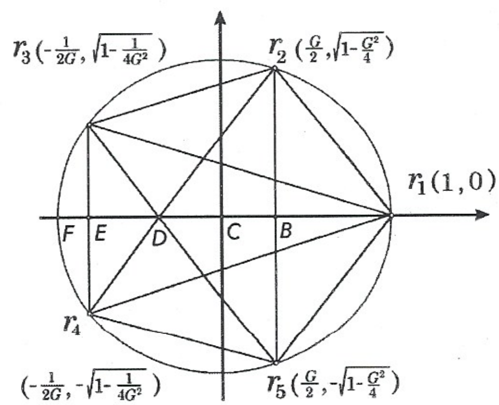


Figure 5 The ratio G in the roots of the equation $x^5 - 1 = 0$.

The expressions for r_3 and r_4 can be simplified through the relation $1 + G = 1/G$ which is derived from the fundamental equation $x^2 + x = 1$ through dividing it by x: $x + 1 = 1/x$. As G is its positive root we have $G + 1 = 1/G$. Therefore

$$r_3 = -\frac{1}{2G} + \sqrt{1 - \frac{1}{4G^2}} \cdot i$$

$$r_4 = -\frac{1}{2G} - \sqrt{1 - \frac{1}{4G^2}} \cdot i$$

Figure 5 shows the location of the five roots on the complex number plane. They lie on the circle with the radius of one unit. The abscissae of the five points are the real parts of the roots:

$$1; \quad \frac{G}{2}; \quad -\frac{1+G}{2} = -\frac{1}{2G}; \quad -\frac{1+G}{2} = -\frac{1}{2G}; \quad \frac{G}{2}$$

and the ordinates the imaginary parts:

$$0, \qquad \sqrt{1 - \frac{G^2}{4}}; \qquad -\sqrt{1 - \frac{G^2}{4}}$$

$$\sqrt{1 - \frac{(1+G)^2}{4}} = \sqrt{1 - \frac{1}{4G^2}};$$

$$-\sqrt{1 - \frac{(1+G)^2}{4}} = -\sqrt{1 - \frac{1}{4G^2}};$$

From the abscissae we obtain the ratios of the line segments in a pentagram along its axis of symmetry to the radius of the circumscribed circle. the are all simple linear functions of G.

$$AB = 1 - \frac{G}{2}; \qquad BC = \frac{G}{2};$$

$$CD = BD - BC = AB - BC = 1 - \frac{G}{2} - \frac{G}{2} = 1 - G.$$

$$EF = 1 - \frac{1+G}{2} = \frac{1}{2} - \frac{G}{2} = \frac{1-G}{2}$$

Also the radius of the circle which is inscribed in the pentagon is a simple linear function of G. The radius equals

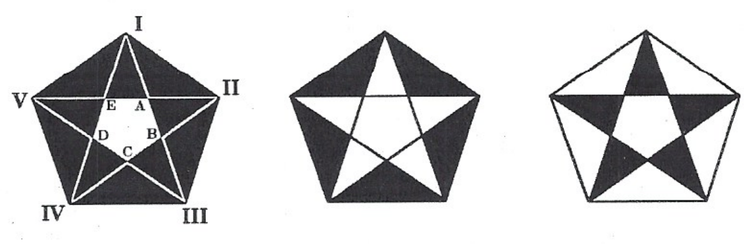
$$EC = \frac{1+G}{2} = \frac{1}{2G}$$

Thus the ratios of the segments of the pentagram to the pentagon sides s being expressed along an axis of symmetry to the radius of the circumscribed circle R being also expressed by G, all that remains is to tie the two groups together. This will be achieved through finding the ratio between s and R. The answer is contained in the ordinate for r_3 in figure 5. R being the radius of the circumscribed circle, half the side of the pentagon is

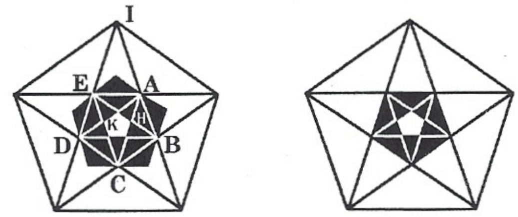
$$\frac{s}{2} = \sqrt{1 - \frac{1}{4G^2}} \cdot R \quad \text{or} \quad \frac{s}{R} = 2\sqrt{1 - \frac{1}{4G^2}}$$

which again expresses itself through G.

The number G is also the ratio of areas which are formed between the pentagon and the pentagrams. The sequence of areas which is marked in the five diagrams of Figure 6



(i) (ii) (iii)



(iv) (v)

Figure 6 Areas forming a geometric progression with the ratio G

constitutes a geometric progression with the ratio G. The ring-shaped area marked in the first diagram, (i), is composed of five times the area of ΔI II E. Taking II E as the base and diminishing it by multiplying with G while keeping the altitude of the triangle unchanged, we get ΔII A I which is one of the five marked triangles of the second diagram, (ii). In comparison to ΔII A I the base of ΔAEI is again reduced by G while its altitude remains the same. five times the triangle AEI equals the marked triangles of the third diagram. In order to take the next step to the fourth diagram, (iv), we consider again ΔAEI which is congruent to ΔAEC. Taking AC as its base and reducing it by the ratio G to CH without changing the altitude we get ΔHCE which is congruent to ΔCED. By subtracting from the triangle CED the triangle EDK and adding instead the congruent triangle CDF we obtain the quadrilateral $CKDF$ which taken five times makes up the marked area of the fourth diagram. This area, therefore, represents the third diagram's area reduced by G. Finally, we take up once more the triangle CED which equals one-fifth of the marked area of the fourth diagram. Reducing its base CE by the ratio G without changing the altitude, we obtain the triangle CKD which taken five times makes up the marked ring-shaped area of the fifth diagram. Denoting the total marked areas of the first diagram with A the marked area of the successive diagrams form the geometric progression: A; AG; AG^2; AG^3; AG^4. The last ring-shaped area occupies the same place within the inner pentagram as the first ring-shaped area in the outer one. The ratio between the two rings is therefore G^4 which checks with a previously found result that corresponding sides of the two pentagons have the ratio G^2. the white area left over in the middle of the last diagram is also G^4 times the white area in the middle of the first diagram.

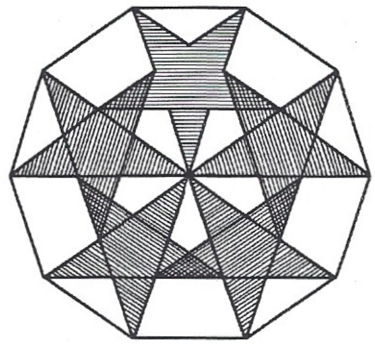


Figure 7 The ratio *G* in the regular decagon.

The part the ratio *G* plays in a pentagram also carries over into the domain of the regular decagon: *G* is the ratio of the side of a regular decagon to the radius of the circumscribed circle. Figure 7 shows a regular decagon. Its vertices are joined with the center and thus the angle of 360° around the center is divided into ten equal angles of 36°. Ten pentagrams can be placed around the center to fit in these spaces. Every second of them is drawn in Figure 7 and marked through shading. The sides of these pentagrams equal the decagon side. the ratio between these two sizes is *G*. The usual construction of the side of a regular decagon to be inscribed in a given circle is an application of *G*.

In solid geometry *G* reappears in the geometry of the pentagon-dodecahedron, which contains pentagons as its faces, and in the icosahedron which contain pentagons as plane sections.

The ratio *G* appears furthermore in geometric figures which are not connected with pentagons or decagons. One of them is a square which is inscribed in a semi-circle (Figure 8). Whereas

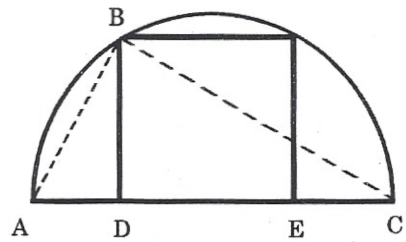


Figure 8 The ratio G in a square inscribed in a semi-circle

the the three line segments of a pentagram side have the smaller one in the middle and the larger ones to the sides, we have the reverse sequence in Figure 8. Nevertheless, the ratio between the two segments is again *G*. To prove it we use the similar triangles *BCD* and *ABD*. Denoting the ratio of the shorter to the longer legs as *x* we have:

$$\frac{BD}{CD} = x; \qquad \frac{AD}{BD} = x$$

and therefore $BD = CD \cdot x$; $AD = BD \cdot x = CD \cdot x^2$. As $AD + BD = AD + DE = AE$ and $AE = CD$ we have: $CDx^2 + CDx = CD$ or $x^2 + x = 1$, the positive root being *G*. This result can also be interpreted in solid geometry, dealing with an equilateral cylinder inscribed in a hemisphere.

Another appearance of *G* occurs in a circle inscribed in an isosceles triangle which in turn is inscribed in a square (shown in Figure 9) or, interpreted in solid geometry, in a sphere inscribed in a cone which in turn is inscribed in an equilateral cylinder or a cube. The three angles in Figure 9 marked as ϕ are equal to one another (one pair are angles on the base of an isosceles triangle, and another pair perpendicular angles) therefore: $\Delta ABE \sim \Delta AED$ (the triangles have one angle in common and contain another pair of equal angles ϕ). Therefore

$$\frac{AB}{AE} = \frac{AE}{AD}.$$

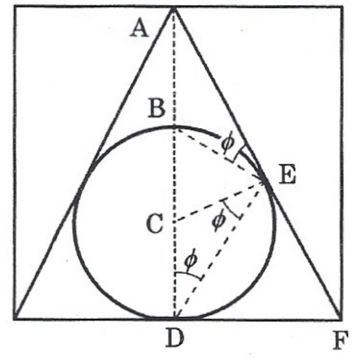


Figure 9 The ratio G in a circle inscribed in an isosceles triangle which is in turn inscribed in a square

Denoting these two, equal ratios as x we have $AE = AD \cdot x$ and $AB = AE \cdot x + AD \cdot x^2$. Then ΔADF ~ AEC (the triangles have both right angles and have their angle at A in common). In the large triangle ADF the ratio of the larger to the smaller leg is 2; therefore the corresponding ratio in the smaller triangle is also 2 and AE must equal twice the radius of the circle. Therefore, $AE = BD$. Substituting AE which is $AD \cdot x$ for BD and $AB = AD \cdot x^2$ for AB into the equation $AB + BD = AD$ we get $ADx^2 + ADx = AD$ which is again the fundamental equation $x^2 + x = 1$ with the positive root $x = G$.

G is also the ratio of the smaller leg to the hypoteneuse of a right triangle the sides of which form a geometrical progression. (The right triangle the sides of which form an arithmetic progression is the Egyptian triangle with the sides 3; 4; 5). Denoting the hypoteneuse of a right triangle whose sides form a geometrical progression as "a" the larger leg is $a \cdot x$ and the smaller leg is $a \cdot x^2$. From the theorem of Pythagoras we get $a^2 = (ax)^2 + (ax^2)^2$ or $x^4 + x^2 = 1$ which gives for x^2 the positive root G. Therefore the smaller leg of the right triangle being $a \cdot x^2$ equals $a \cdot G$.

Arithmetically the number G also shows outstanding qualities. First it has the same infinite sequence of decimals as its reciprocal value: $G = 0.61803398875 \cdots$.

$$1/G = 1.61803398875 \cdots.$$

It is the only positive number which forms its reciprocal value by adding 1. This results from the equation $x^2 + x = 1$ by dividing it by x: $x + 1 = (1/x)$. Then G can be expressed as the limit of a continued fraction written only using the numeral 1.

$$G = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$$

By computing this continued fraction step by step, we get the following fractions:

$$1; \frac{1}{2}; \frac{2}{3}; \frac{3}{5}; \frac{5}{8}; \frac{8}{13}; \frac{13}{21}; \frac{21}{34}; \frac{34}{55}; \frac{55}{89}; \frac{89}{144}; \cdots.$$

The numerators can be obtained by adding the numerators of the two preceeding fractions, and the same holds good for the denominators form a Series of Fibonacci:

$$1; 1; 2; 3; 5; 8; 13; 21; 34; 34; 55; 89; \cdots$$

in which each term is the sum of the two preceeding ones. G is the limit of the ratios of two successive terms in the Series of Fibonacci. This series starts with two terms of 1. If instead any other numbers are chosen (excluding zero) which can be integers or fractions and the same procedure is applied to them G will still appear as the limit of the ratios of two successive terms. This is shown in the following example in which the arbitrarily the numbers 5 and 24 have been chosen:

=	5			
=	24	5 ÷	24	= 0.2083...

5	+	24 =	29	24 ÷	29	= 0.8276...
24	+	29 =	53	29 ÷	53	= 0.5472...
29	+	53 =	82	53 ÷	82	= 0.6463...
53	+	82 =	135	82 ÷	135	= 0.6074...
82	+	135 =	217	135 ÷	217	= 0.6221...
135	+	217 =	352	217 ÷	352	= 0.6165...
217	+	352 =	569	352 ÷	569	= 0.6187...
352	+	569 =	921	569 ÷	921	= 0.6178...
569	+	921 =	1490	921 ÷	1490	= 0.6181...
921	+	1490 =	2411	1490 ÷	2411	= 0.6180...
1490	+	2411 =	3901	2411 ÷	3901	= 0.6180...
. . .		. . .	. .	. . .	.	. . . . . .

In our example, the first four decimals of G are obtained at the 11^{th} division

G can also be expressed as a limit of square roots in which 1 is again the only numeral used:

$$G = \cfrac{1}{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}$$

(the proofs which use the theory of limits are here omitted).

In the history of mathematics references to the number G lead back to the oldest geometric records. There is a passage in Herodotus in which he relates that the Egyptian priests had told him that the proportions of the Great Pyramid at Gizeh were so chosen that the areaof a square whose sides is the height of the Great Pyramid equals the area of a face triangle. Writing "$2b$" for the side of the base of the Pyramid (see Figure 10) and "a" for the altitude of a face triangle and "h" for the height of the Pyramid, Herodotus' relation is expressed in the following equation:

$$h^2 = (2b \cdot a)/2 = a \cdot b$$

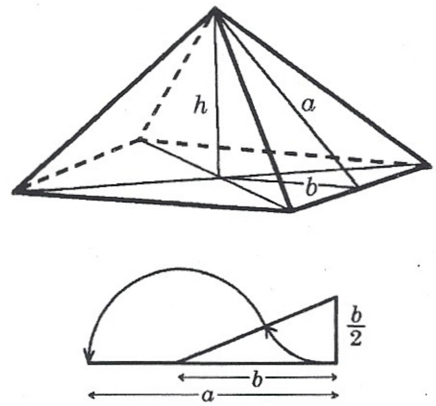


Figure 10A

As "a" is the hypoteneuse of a right triangle with the legs "b" and "h" we can apply the theorem of Pythagoras and get: $a^2 = b^2 + h^2$ or $h^2 = a^2 - b^2$. Equating the expressions for h^2 in the two equations we obtain $a^2 - b^2 = ab$ or $b^2 = ab = a^2$. Dividing the equation by a^2 we have $(b/a^2) + (b/a) = 1$. substituting x for the ratio b/a we are back at the equation $x^2 + x = 1$ which has G as its positive root. Therefore, G is the ratio of half the side of the base square of the Great Pyramid to the altitude of the face triangle. Checking with the actual measurements taken at the Great Pyramid we have:

$h = 148.2$ m. (reconstructed height of undamaged apex)
$b = 116.4$ m.

which makes

$$a = \sqrt{148.2^2 + 116.4^2} = 188.4$$

and gives the ratio $b/a = 0.6178 \cdots$. Comparing with $G = 0.6180$ the difference is $0.0002 \cdots$.

A further consequence of the statement of Herodotus is the fact that G also appears as the ratio of the base to the lateral area of the Great Pyramid. The sum of the areas of the four face triangles of the Great Pyramid is $4\cdot(2a\cdot b)/2 = 4ab$. The ratio of the areas is therefore

$$\frac{4b^2}{4ab} = \frac{b}{a} = G$$

The ratio G can be used to construct the form of the Great Pyramid. In Figure 10 first the ground plan of the Pyramid has been drawn. It is a square with its diagonals. Then the elevation is drawn with the positions of the the base vertices

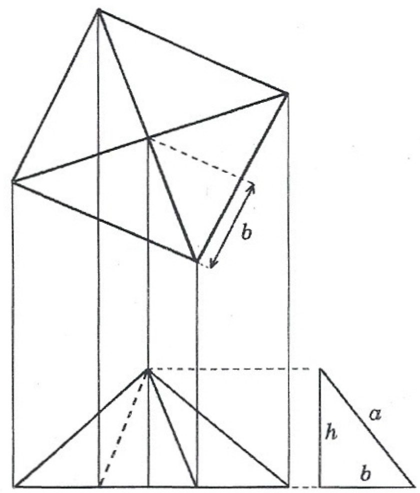


Figure 10B Construction of the form of the great pyramid

determined through vertical lines dropped down from the corresponding points of the ground plan. What remains to be drawn is the height of the Pyramid. The altitude of a lateral face is $(1/G)\cdot b$ repeating the construction for G as described before using b as a base, one obtains $b\cdot G$. Adding $b\cdot G$ to b furnishes $b(G + 1) = b(1/G) = a$. Using a as the hypoteneuse and b as one leg of a right triangle, the length of the second leg is the height of the pyramid h. thus the elevation can be completed. The third projection (88888) has been obtained from the ground plan and elevation through the methods of descriptive geometry, (described in the Mathematics Teacher, April 1946).

In the sixteenth century (1509) Paciolo di Borgo wrote his treatise *DeDivina Proportione* (Of the Divine Proportion) on the ratio *G*. Kepler refers to it as the *sectio divina* (divine section) and Leonardo da Vinci as *sectio aurea* (the golden section) which is a term still in use for it. In an extensive literature on The Golden Section numerous facts have been collected which show its appearance in the forms of nature and art. Hambridge based on it his aesthetic research on *Dynamic Symmetry*. Kepler whose sense of proportional relations led him to his three astronomical laws which are the starting point of modern astronomy speaks of the properties of *G* in his "Mysterium Cosmographicum de Admirable Proportione Orbium Celestium" as of those of one of the two "great treasures" of geometry, the second being the Theorem of Pythagoras.

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Kepler's Solid
and Other Construction Puzzles

Gerhard Kowalewski

Forward

The main object of this booklet is to build the Kepler solid that is bounded by thirty plane rhombi; it can be found in the polyhedra studies of the great astronomer. It is contructed out of two sets of multi-colored blocks having ten blocks in each set, which are joined together along like colors. They are similar to MacMahon's cubes. This gives a new kind of branching game that is associated with Kepler's thirty sided solid; we hope it will attract interest because of its difficulty. A person who would play successfully without knowing the theory would have to be lucky.

There is a connection between Kepler's thirty sided solid and a construction in 6 dimensions which is made up of squares and which can be taken as the prototype of the Kepler polyhedron: we are convinced that Kepler would have been filled with enthusiasm to have known that this 6 dimensional cube shelters two such Kepler polyhedra within it.

I want to express my debt to my brother, the Königsberg philosopher, an excellent scholar in the theory of color, for his many valuable suggestions. I was able to speak with him during his visit to Dresden in the Summer of 1937 about the details concerning the publication of this booklet.

As with the previous booklets of the series that I have established, the aim is to reach for a general understanding of the subject.

Dresden, White Stag, Winter 1937/38

Gerhard Kowalewski

CHAPTER ONE

Constructions with Multi-Colored Squares and Cubes

A square is divided by its diagonals into four right angled triangles; let us color them with four different colors (red, yellow, green, and blue, for example). These colors are indicated in the Figure by *1*, *2*, *3*, *4*; there are six arrangements of colors, counted so that those distributions of color which amount to a mere rotation of the square are not to be counted as distinct.

Figure 2 shows the six possible arrangements of the colors *1*, *2*, *3*, *4*.

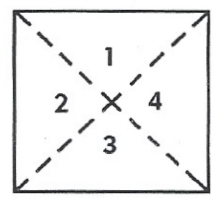

Figure 1

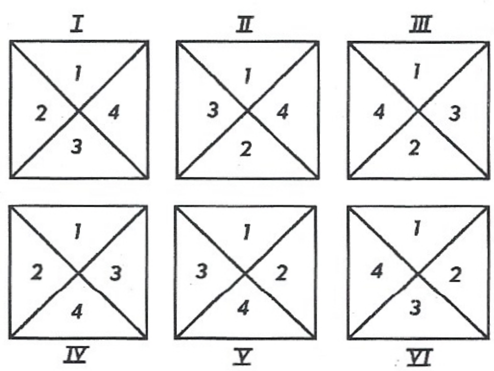

Figure 2

If we had at our disposal a single specimen of each of these six multicolored tiles, then we could take up the following problem.

Build one big square out of four of these multi-colored squares so that the large square has matching colors along the top, bottom, right, and left sides. For example, we might duplicate the edge coloring of Figure 1. The tiles should be layed so that they obey what we shall call the *domino rule*, that is, squares should only be joined along edges of the same color.

The solution of the problem is easily obtained by looking at Figure 3, where the unknown colors are indicated with the letters *a*, *b*, *c*, *d*.

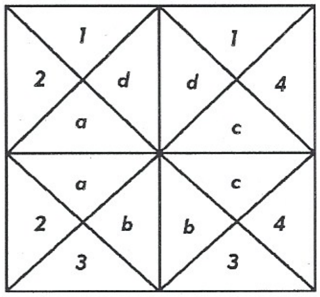

Figure 3

It helps here to use the old mathematical trick of naming the unknown. In the lower left hand tile *a* appears with 2 and 3, so it must be different from them: but also a glance at the upper left tile shows that it must be different from *1* and 2. So *a* can only be the color *4*. For similar reasons *b* must be the color *1*, *c* must be 2, and *d* can only be 3, so that the whole construction looks like what is shown in Figure 4. This procedure produces a big square having the same boundry colors as those of Figure 1, using the four tiles displayed in Figure 5. Colors which stand opposite each other in Figure 1 are neighbors in each tile of Figure 5. This observation fully determines the choice of the four tiles.

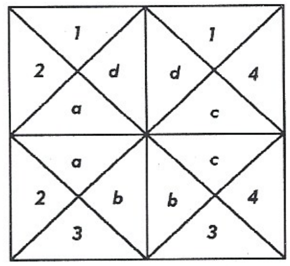

Figure 4

By turning them so that *1* is at the top, 3 will have to be either on the right or the left. Either choice for the placement of 3 yields two choices for the placement of *2* and *4*. With the same four tiles one can obtain a big square, obeying the domino principle, which shows the same edge colors as square VI in Figure 2.

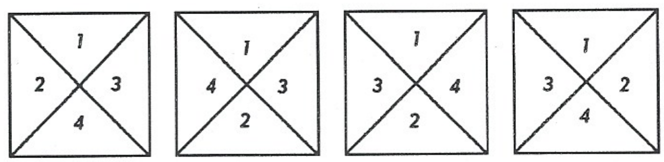

Figure 5

Instead of coloring the quarter squares , one could just as well color only along the sides of the square, framing them in color. The domino principle requires that only edges of like color may be joined together.

MacMahon's building blocks imitate these problems but in three dimensional space. In place of squares there are now cubes whose faces are painted in six different colors. We shall represent these colors by *1*,..., 6; and as before we shall regard two arrangements the same if they can be obtained from each other merely by rotating the cube. Now, how many different color arrangements are there? Suppose that the bottom face, upon which the cube is standing, is colored with the color *1*. One of the colors 2,..., 6 must show on top. This gives five different possibilities. These five types depend on whether *1* and 2; *1* and 3; *1* and 4; *1* and 5; or *1* and 6, are opposite

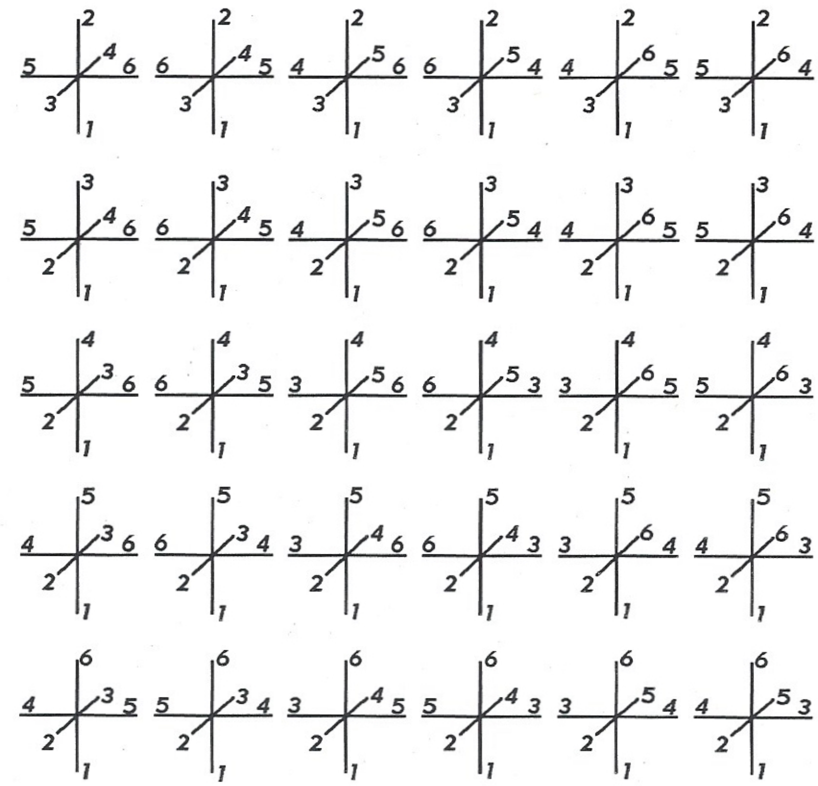

Figure 6

each other. In the type in which *1* and *a* are found on the top and bottom respectively, one can always turn the cube so that the color *b* comes to the front. That leaves three possibilities for the color on the back side; and the two remaining colors can be used in two ways (left and right, or right and left). Each of the five types therefore has six multicolored cubes. The total number of MacMahon cubes is 30, while there were only 6 different kinds of bordered squares.

In Figure 6 the 30 MacMahon cubes are shown schematically as crossed axes. Each of the six arms of these crossed axes reaches from the mid-point of the cube to the middle of a cube face and is marked with the color number of the cube face in its direction.

MacMahon's problem is this: build a big cube using eight of his thirty cubes while adhering to the domino principle. This big cube is to show a prescribed color distribution agreeing with that of a model cube that was previously chosen.

If the pattern cube is, for instance, the first one shown in Figure 6, then the eight cubes must show a pattern which can be exploded as in Figure 7 to make it easier to see.

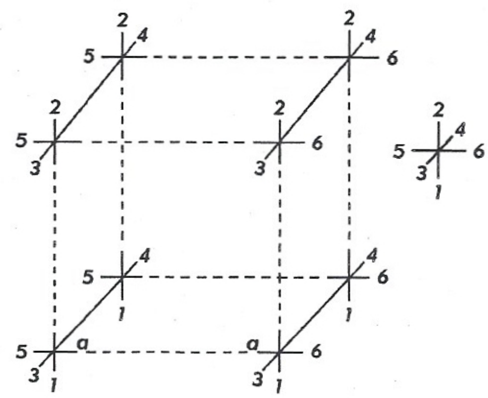

Figure 7

The color *a* in Figure 7 must be different from *1*, *3*, *5* and also *1*, *3*, *6*. So there remain only the possibilities $a = 2$ and $a = 4$. When one of these two possibilities is chosen, the remaining colors are obviously fixed and one comes to Figure 8 and Figure 9.

To make it easier for the reader to check on his own drawing we have shown the order in which the absent colors are found, through the use of subscripts. For example, in Figure 8 it is 6_1 that is found first through the fact that this color must be different from *1*, *4*, and *5*. Next 4_2 is found through the fact that among *1*, *2*, *3*, *5*, *6* only *4* is missing, and so on.

MacMahon's problem has two solutions (Figures 8 and 9). By close observation one finds that in both cases the same

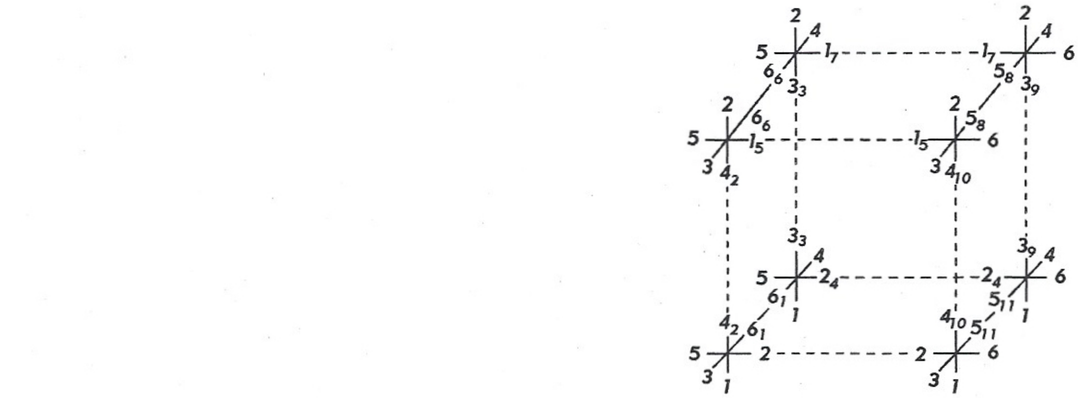

Figure 8

eight cubes participate. The four cubes on the upper story of one figure build the first floor of the other.

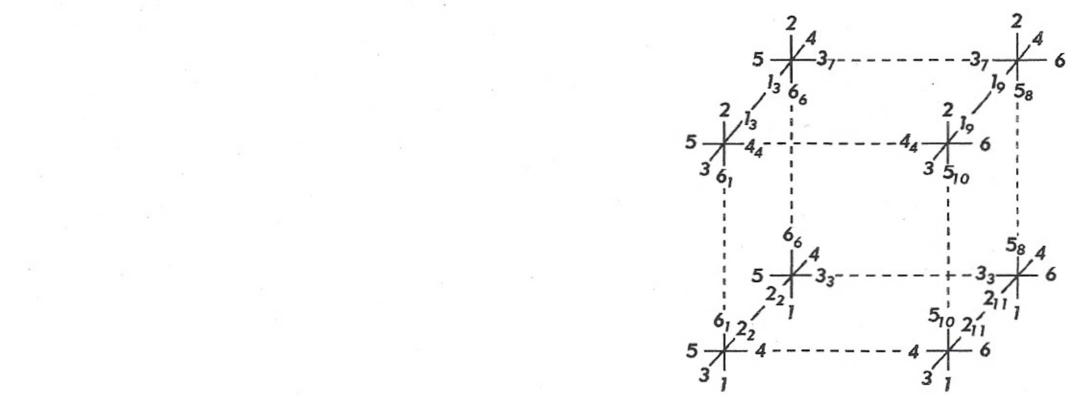

Figure 9

I have proposed using a form of the crossed axes which are quite handy for building. Compared to MacMahon's blocks the use of crossed axes has the advantage that obedience to the domino principle is easy to control. I glue six cubes of the same size onto the faces colored *1*,..., 6 of the MacMahon cube; they are the same size as the cube face to which they have been attached. Figure 10 shows such a cross in which, for the sake of clarity, the front and back arms are left off and only suggested by extended edges.

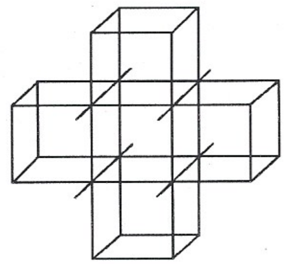

Figure 10

The first specimen of this very decorative toy was lost when I sent it to a trade fair. It seems that the thirty multi-colored crosses could not resist publicity and entered into the toy trade quite on their own. There even appeared, in an illustrated magazine, a picture of Shirley Temple playing with them.

For further information about MacMahon's blocks one can refer to my book *Alte und neue mathematische Spiele* and also to Ferdinand Winter's beautiful monograph on the multi-colored cubes, both published by B.G. Teubner. The mathematical, particularly the group theoretic, aspect of MacMahon's problems have been treated by my ingenious student Walter Stams in the latest edition of *Deutschen Mathematik.*

There is a plane analog of my crossed axes of glued together cubes; it is shown in Figure 11. The central square is mirrored in all four sides to produce a cross.

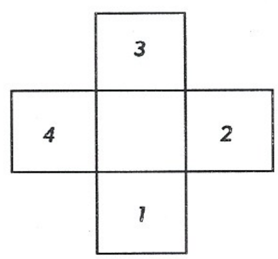

Figure 11

CHAPTER TWO

Rhombic Skirts for the Platonic Solids

The five Platonic solids – tetrahedron, cube, octahedron, dodecahedron, icosahedron – were investigated by the ancient Greek mathematicians and constitute the subject of Euclid's *Elements*.

These five solids acquired an unanticipated significance in Kepler's *Mysterium Cosmographicum* (1596). he believed that with their help he could establish a law of planetary distances for the planets known at that time: Saturn, Jupiter, Mars, Earth, Venus, mercury. The astronomers associated a sphere with each of these planets; it was centered at the sun and passed through its respective planet. One might wonder whether or not the planets keep the same distance from the sun, an idea that Kepler himself settled by discovering that the planetary paths form an ellipse with the sun at one focus. Kepler found in 1596 that the five Platonic solids could be constructed between the six planetary spheres, so that each such solid has its circumscribed and inscribed planetary sphere. The specific sequence is: Saturn, (cube), Jupiter, (tetrahedron), Mars, (dodecahedron), Earth, (icosahedron), Venus, (octahedron), Mercury. Later when Uranus and Neptune entered into the circle of planets this series attracted less and less interest.

Kepler's planetary construction was very fruitful in stimulating his geometrical research. He did extensive work in the field of polyhedra studies and discovered, for example, the star polyhedra. He had another very good idea too. He created what I call the rhombic skirts for the five Platonic solids. Let us fix the idea of this construction using the example of the cube.

From directly above the middle of each of the six faces of the cube one can drop a perpendicular to the cube edges having some length, say h. Next, consider the cube edge AB. There are two cube faces that meet there. Connect A and B to the mountain peaks E, F situated along the perpendiculars. Doing this creates triangles EAB and FAB. If h is given the appropriate value ($h = a/2$), the triangles will lie in a single plane and will form a rhombus braced by AB on the diagonal. When we construct the rhombus that belongs to each one of the twelve edges of the cube, then we have a rhombic dodecahedron: We shall say that it is obtained by dressing the cube in its rhombic skirts.

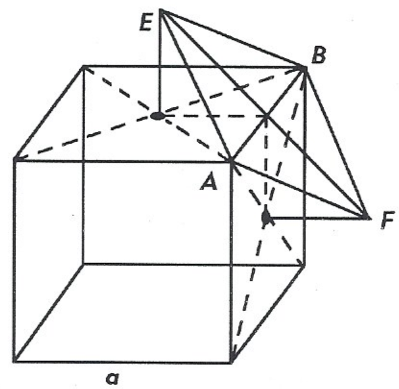

Figure 12

It is obvious in Figure 12 that $EF = a\sqrt{2}$, $AB = a$, so that the edges of the original cube are the short diagonals of the rhombic dodecahedron. The long diagonals form the edges of an octahedron. The same rhombic dodecahedron serves as skirts both for a cube and an octahedron.

In the same way, the rhombic skirts of an icosahedron and those of a dodecahedron are the same things too. These skirts fit together as thirty rhombi making Kepler's celebrated triacontahedron; we call it *Kepler's solid*. We will see later that the diagonals of the rhombi introduced here are related to each other in a golden section, that is they stand in the ratio

$$\tfrac{1}{2}(\sqrt{5} - 1) \text{ to } 1.$$

The triacontahedron was of extraordinary interest to Kepler because two of them hold the earth sphere in his planetary construction. The short diagonals of the thirty rhombi of Kepler's solid are the edges of a dodecahedron and the longer diagonals are those of an icosahedron. Speculation about the golden section, which Kepler called the *divine section*, exercised a special charm for investigators who were predisposed to mysticism. The ancient Greek mathematician Eudoxus (400 to 350 B.C.) was the first to divide a segment according to the golden section. One makes use of this division when, for example, constructing a regular decagon. To find the edge length of an inscribed decagon one needs to divide the radius of a circle in a golden section. The larger piece gives the edge of the decagon. The golden section is that division of a line segment such that the whole piece is related to the larger part as the larger part itself is to the smaller. Taking the whole segment as unity and as the larger part, the smaller part will be $1 - z$, so the following proportion holds:

$$1 : z = z : 1 - z.$$

From this it follows that $z^2 = 1 - z$ or $z^2 = z + 1$, giving $(z + ½)^2 = 5/4$, that is to say $z = ½(\sqrt{5} - 1)$. This is therefore the length of the larger part of a golden section division when the entire segment has unit length. If we develop the irrational number $½(\sqrt{5} - 1)$ a continued fraction, we write using $z^2 + z = 1$.

$$z = \frac{1}{1+z} = \frac{1}{1+\frac{1}{1+z}} = \frac{1}{1+\frac{1}{1+\frac{1}{1+z}}} = \ldots$$

From this it follows that z the irrational number of the golden section, can be represented by the infinite continued fraction:

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}$$

which is the simplest possible continued fraction because it involves only the number 1.

When one describes a circle of radius 1, then the continued fraction given above will represent the side of an inscribed, regular decagon. The sequence of convergants:

$$\frac{1}{1}, \qquad \frac{1}{1+\frac{1}{1}}, \qquad \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$$

$$\text{or} \quad 1, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{5}, \ldots$$

shows the following rule which is based on the general principles of the theory of continued fractions: In each fraction the numerator is the sum of the two preceding numerators and the denominator is the sum of the two preceding denominators. For example, 2 = 1 + 1, 3+ 2 + 1, and 3 = 2 + 1, 5 = 3 + 2, and so on. On the basis of this rule one can easily calculate the sequence of convergants. After 1, 1/2, 3/5, 8/13, ... follow 5/8, 8/13, 13/21,... The number z lies between any two successive terms of this sequence. The fractions of odd index

$$1, \ \frac{2}{3}, \ \frac{5}{8}, \ldots,$$

converge from below while those of even index

$$\frac{1}{2}, \ \frac{3}{5}, \ \frac{8}{13}, \ldots,$$

converge from above. It is a peculiar feature of the sequence

$$1, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{5}, \ \frac{5}{8}, \ ,\frac{8}{13}$$

that each denominator appears as the numerator of the next term of the sequence. first you calculate the denominators q_1, q_2, q_3, ... by putting $q_1 = 1$, $q_2 = 2$ and then using the rule $q_n = q_{n-1} + q_{n-2}$, once that is done the numerators p_1, p_2, p_3, ... of the fractions

$$1, \ \frac{}{2}, \ \frac{}{3}, \ \frac{}{5}, \ \frac{}{8}, \ \frac{}{13}, \ldots$$

can immediately be written down using the rule $p_n = q_{n-1}$. This gives the fractions 1, 1/2, 2/3, 3/5, 5/8, 8/13, ...

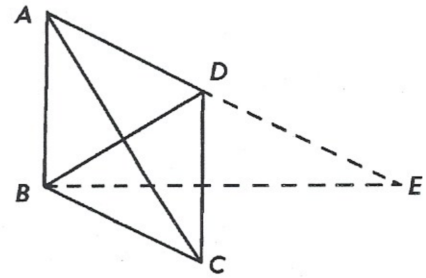

Figure 13

In Figure 13 a Kepler rhombus is shown, that is to say a rhombus whose diagonals *BD* and *AD* stand in a golden section relationship to each other like the side of a regular decagon to the radius of its circumscribed circle. If one were to draw a circle having *AC* as its radius, then the span *BD* would go around it exactly ten times. We will now derive a property of the sharper of the two angles in the Kepler rhombus. Obviously

$$\tan(\alpha / 2) = z$$

where z is the number $\frac{1}{2}(\sqrt{5} - 1)$. From this follows

$$\tan\alpha = \frac{2\tan(\alpha / 2)}{1 - \tan^2(\alpha / 2)} = \frac{2z}{1 - z^2}$$

Because $z^2 + z = 1$, we also must have that $1 - z^2 = z$, thus

$$\tan(\alpha) = 2.$$

To construct a Kepler rhombus having a given side *AB* erect a perpendicular to *AB*, say *BE*, having double the length of *AB* (Figure 13). Since $\tan(\alpha) = 2$, *A* will remain an angle of *ABE* of measure a. Now make *AD* = *AB* and swing two circles about *A* and *B* with a radius *AB*, these will intersect at A and also at a new point *C*. The Kepler rhombus that we are looking for is *ABCD*.

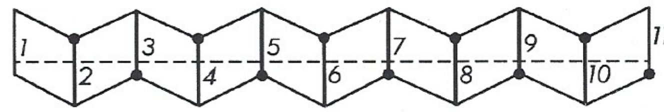

Figure 14

It is vitally important that the reader have a cardboard model of Kepler's solid at his disposal. You can easily produce one for yourself. In Figure 14 there are ten, articulated Kepler rhombi that can be drawn on cardboard. You must cut out the figures and score the edges that are shown as dark lines, in order to fold them more easily. It is to be done so that in the finished position the points *1, 2, 3, ..., 10, 1* form a regular decagon. Then the Kepler rhombi are inserted at the points marked with heavy dots. Finally we must attach a cap made of five Kepler rhombi to the top and bottom to close the openings. This finishes the construction of Kepler's tricontahedron. The reader should interrupt his reading at this point in order to finish the model without being hasty in the work.

Let us add a remark concerning the rhombic skirts of the tetrahedron. It can be made plain from Figure 15 that this gives a cube. The rhombi in this case turn out to be squares.

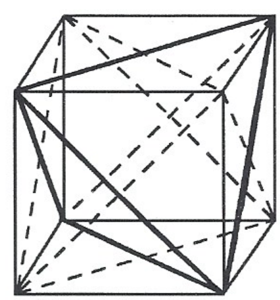

Figure 15

The same cube could equally well be taken as the rhombic skirts of a second tetrahedron shown by dashed lines in Figure 15, and can be said to be the opposite of the one drawn with solid lines.

Coloring Kepler's Solid

We will suppose that the reader has the cardboard model that we have urged him to construct. If you place it on one of its thirty rhombi, then at the very top there is a rhombus that is horizontal and directly above the base rhombus. If we then turn the model so that the long diagonals of these two rhombi are running from front to back, the shorter ones running left to right, then one of the thirty rhombi stands in front, one on the very back, another on the extreme right, and one on the extreme left. The planes of these four rhombi, along with those of the top and bottom ones, form a cube with the base, as before. Label these six rhombi all with the number 2. Proceeding in this way until we reach the number 5, all thirty rhombic faces of Kepler's triacontahedron become labeled with numbers. The six rhombi that make up a cube have the same number.

Now the reader should obtain five colored sheets of paper. Cut six Kepler rhombi that are of the same size as those on the model of the Kepler solid out of each sheet. Glue a red piece, for example, onto all faces with the label *1*; those numbered 2 get, say, yellow pieces glued onto them; those numbered 3 get green; all with a *4* get blue; finally make the ones numbered 5 white. In this way the Kepler solid gets colored with five colors and presents a surprisingly beautiful appearance.

The Kepler solid has, resulting from its skirt relationship to the dodecahedron and the icosahedron, 20 corners with three edges, three way corners, and 12 of the five-way corners. Surrounding each of these five-way corners the five colors *1*, ..., 5 are arranged in a specific way. Reading clockwise beginning with white you get 12 different arrangements of the other colors, and thus of the numbers *1*, ..., *4*. On the model that is here in front of me, I can determine the following 12 different arrangements:

1 2 3 4	*2 3 4 1*	*3 2 1 4*	*4 1 2 3*
1 3 2 4	*2 4 1 3*	*3 1 4 2*	*4 2 3 1*
1 2 4 3	*2 1 3 4*	*3 4 2 1*	*4 3 1 2*

Each of them arises from the numbers 1, 2, 3, 4 by an even number of transpositions, that is, exchanging the places of some pair of numbers. If one interchanges two of the numbers within the twelve arrangements, 1 and 2 for example, then the twelve completely new orderings arise which, disregarding changes in the listing, are as follows:

1432	2341	3214	4123
1324	2413	3142	4231
1243	2134	3421	4312

This now exhausts all 24 arrangements of *1, 2, 3, 4*. One can conclude from this observation that Kepler's triacontahedron can be colored in two and only two ways with the colors *1*,..., 5 so that the six rhombi belonging to the same cube share the same color. A coloring that can be made to agree with another by a rotation are not taken as different. This fact was already observed by my brother in his work on color arrangements in the *Berichten der Wiener Akademie* 11, May 1916.

It is very attractive to see both of the colorings together in front of you. One would have to create two copies of Kepler's solid. If the reader is willing to take the trouble, he would have acquired an attractive desk top ornament and a valuable model to illustrate group theoretic relationships: all you would then need is a friendly mathematician to teach you some group theory.

Each of the 20 three-way corners of the Kepler triacontahedron are colored with three colors *a*, *b*, *c*. There are ten of these triads showing the five colors. Each triad appears twice on opposite three-way corners but the opposite triads have their colors in different cyclic arrangements as in :

a *b*
c

a *c*
b

Supplementary Geometric Observations

We still have to supply proof that the diagonals of a Kepler rhombus stand in a golden section ratio to each other. In Figure 16 two neighboring triangles of an icosahedron are shown. *ACB* is half of a Kepler rhombus based on *AB*.

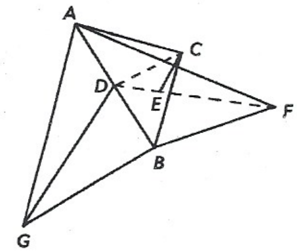

Figure 16

The rhombus forms an angle α with both triangles so the complete angle *GDF* is 2α, leaving a supplement of 2β needed to make two right angles. That is $\alpha + \beta = \pi/2$. When *CE* is the perpendicular from *C* to *ABF*, then *E* will be the orthocenter, giving

$$DE = \frac{a\sqrt{3}}{6} = \frac{a}{\sqrt{3}},$$

where a is the side of the triangle. It follows from Figure 16 that

$$DC = \frac{DE}{\cos\alpha} = \frac{DE}{\sin\beta}.$$

The following relationship holds for the diagonal relation of the Kepler rhombus

$$\frac{DC}{DA} = \frac{1}{\sqrt{3}\sin\beta}$$

From this we see that β is half of the angle between two neighboring planes of an icosahedron. To help in the calculation of β one should now place a sphere of radius 1 about one corner of the icosahedron. The five planes of the icosahedron cut out a spherical pentagon whose sides equal π/3 and whose angles equal 2β (see Figure 17).

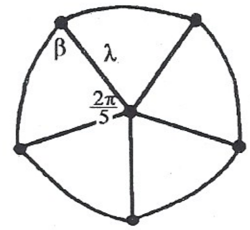

Figure 17

It can be read from Figure 17 that:

$$\cos(^{\pi}/_{3}) = {^1}/_2 \qquad \sin(^{\pi}/_{3}) = {^1}/_2\sqrt{3}$$

$${^1}/_2 = \cos^2\lambda + \sin^2\lambda\cos(^{2\pi}/_{5})$$

$$\frac{\sin\beta}{\sin(2\pi/5)} = \frac{\sin\lambda}{\sqrt{3}/2}$$

The first equation now gives

$$\sin^2\beta = 3(1 - \cos(2\pi/5) = 2/3(1 - \cos(2\pi/5)) = 4/3 \cos2(\pi/5),$$

$$\text{thus} \qquad \sin\beta = \frac{2}{\sqrt{3}}\cos(\pi/5).$$

From this it follows that $\dfrac{DC}{DA} = \dfrac{1}{2\cos(\pi/5)}$.

In Figure 18, $\cos(\pi/5)$ is marked with the long bracket. We can see that $\cos(\pi/5) = ½(1 + z)$. In view of the fact that $z(1 + z) = 1$, we obtain $2\cos(\pi/5) = 1/z$.

$$\frac{DC}{DA} = z = ½(\sqrt{5} - 1).$$

With that we have obtained the desired result. It would have been possible to do so without the help of spherical trigonometry. We suppose that most readers are familiar with the law of spherical sines and cosines. If that is not the case I recommend that you look it up in my book *Lehrbuch der hoheren Mathematik* published by Walter de Gruyter, 1933.

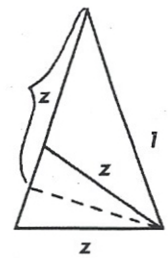

Figure 18

The famous psychologist Fechner, founder of experimental psychology, concerned himself with the aesthetic properties of the golden section and he has remarked that the most attractive rectangle has its sides approximately in a golden section section ratio: They are nearly $\frac{1}{2}(\sqrt{5}-1)$ to 1. If you separate a square off from such a rectangle than a rectangle remains rectangle that is similar to the original. If we regard the long side of the rectangle to be of unit measure, then the short side is z. After removing the square the remaining rectangle has sides of $1-z$ and z, and indeed $1 : z :: z : 1-z$.

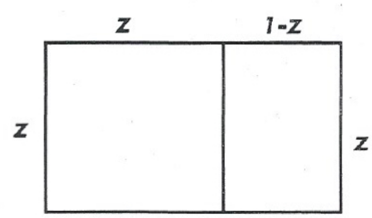

Figure 19

Figure 20 shows another remarkable property of the golden rectangle, that is a rectangle whose sides are in the ratio $z : 1$. Drop perpendiculars from two oppositely positioned corners onto the diagonal that connects the other pair of opposite

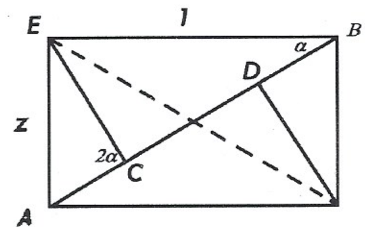

Figure 20

corners. These two perpendiculars are just as long as the diagonal segment between the feet of the perpendiculars.

So we get that $AC{\cdot}AB = z^2$, $CB{\cdot}AB = 1$. Making use of the fact that $AB = \sqrt{1+z^2}$, we obtain:

$$AC = \frac{z^2}{\sqrt{1+z^2}} \qquad CB = \frac{1}{\sqrt{1+z^2}}$$

This gives $CD = AD - AC = CB - AC = \dfrac{1-z^2}{\sqrt{1+z^2}}$

Furthermore $EC^2 + AC{\cdot}BC$, thus $EC = \dfrac{z}{1+z^2}$.

It follows from $1 - z^2 = z$ that $EC = CD$. One might also appeal to the fact, manifest in Figure 20, that $\tan\alpha = z$ and that

$$\tan 2\alpha = \frac{EC}{\tfrac{1}{2}C}.$$

Now make use of the fact that $\tan 2\alpha = 2$ to conclude that $EC = CD$.

The golden section appears in the regular pentagon as shown in Figure 21.

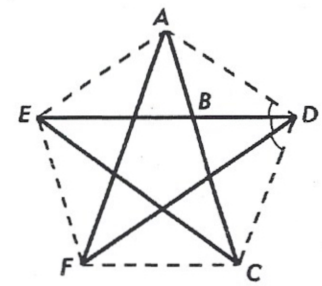

Figure 21

The three angles marked with a stroke are all equal to $\pi/5$; using the theorem on inscribed angles we obtain

$$ABD : BCD = AB : BC.$$

On the other hand

$$ABD : BCD = AD{\cdot}BD\sin(\pi/5) : BD{\cdot}CD\sin(2\pi/5) = 1{:}2\cos(\pi/5) = z : 1,$$

from which it follows that $AB : BC = z : 1$. This means that the diagonal AC is cut in a golden section ratio by the diagonal DE, and also naturally by DF. This gives that $AD : EC = AB : BC$, so we have that $AD : EC = z : 1$. The sides and the diagonals of regular pentagons stand in a golden section ratio. From this it follows incidentally that the angle between the diagonals of a pentagon, like the angle AFD of the figure is equal to $\pi/5$.

Biographical Notes
Gerhard Kowalewski

CHAPTER THREE

The Building Stone of Kepler's Solids

If you examine the Kepler solid in five colors, then certain specific building stones that are contained in it stand out on their own. All you have to do is look at a corner at which three edges meet to recognize a *stubby parallelepiped* that is embedded in the Kepler solid. Three rhombic faces of this parallelepiped meet at the three-sided corner at their obtuse angles, so that they present the appearance that is given here.

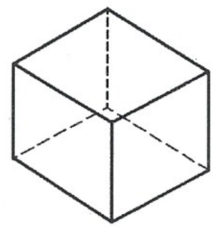

Figure 22

We have indicated the back sides by dashed lines in Figure 22, so that the diagram shows the parallelepiped as it appears. The colored rhombic faces of the parallelepiped that are on the outside surface of the Kepler solid show three different colors. We will use the same color for opposite rhombi, so that the whole block is colored. It will have three of the five colors *1*, *2*, ..., *5* distributed among its rhombic faces so that opposite faces agree. Ten triads can be formed from *1*, *2*, ..., *5* – they are listed below – so there must be ten distinct colored blocks of this kind.

1 2 3,	*1 2 4,*	*1 2 5,*	*1 3 4,*	*1 3 5,*
1 4 5,	*2 3 4,*	*2 3 5*	*2 4 5,*	*3 4 5.*

These ten stubby blocks are, as will be shown, building stones for the Kepler solid.

Besides these, there exist ten other parallelepiped shaped building stones that do not stand out as clearly in the completed model. These have a pair of opposite corners in which the acute angles of the same rhombi meet. We call these building stones the *steep blocks*. They will be colored in the same way as the stubby blocks.

We will see that the Kepler solid can be built up out of these 20 colorful pieces, the ten stubby blocks and the ten steep ones, under the rules of the domino principle, so that the outside surface shows the same pattern of colors as the original. It cannot be shown that Kepler himself recognized these facts in detail. But we have to assume that he had these 20 building stones out of which the Kepler solid can be built. The coloring and the use of the domino principle came later.

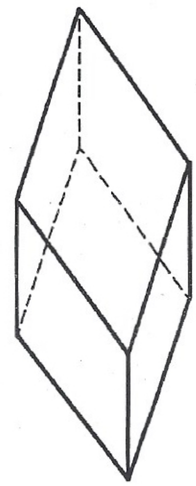

Figure 23

The reader must now interrupt his reading again to make these 20 building stones out of cardboard with colored paper glued on. The construction of the Kepler rhombi has already been explained. The acute angle of such a rhombus fits into a right triangle whose adjacent side is a and whose opposite side is $2a$. This property is the basis of the construction. If you make this angle the vertex angle of an isosceles triangle, you have half of the Kepler rhombus and you can obtain the fourth corner by swinging arcs at the base

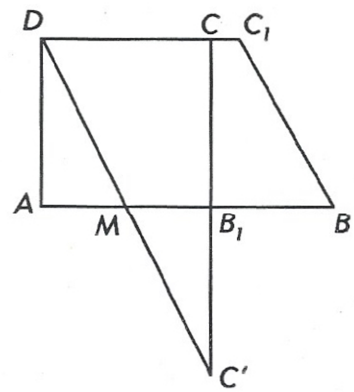

Figure 24

vertices of the isosceles triangle whose radius is the length of one of the legs.

Here is a different way to proceed with the construction. Extend one side of the square *ABCD*, say *CB*, until its length is doubled (Fig. 24). If you draw *DC'* and make $DC_1 = DM$, then DMB_1C_1 is a Kepler rhombus. It is good to do the construction carefully so that the stubby and steep blocks do not have gaps between each other when you build up the Kepler solid out of them.

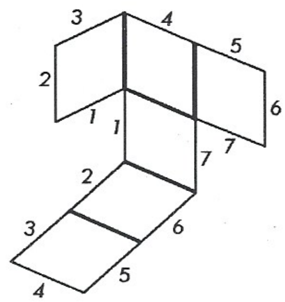

Figure 25a

If the reader has never glued cardboard models and does not know how to make a parallelepiped, he can examine Figures 25a and 25b. The first one gives a net for the steep blocks and the second one a net for the stubby blocks. These nets have to be drawn on cardboard, cut out, scored and then folded along the dark lines. Finally, edges having the same numbers are joined and glued with tabs.

However interesting the puzzles that arise with these blocks, we do not want to take the space here for them. Besides, it is not at all bad to leave the reader with some things to work out for himself.

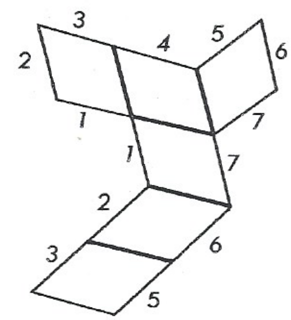

Figure 25b

How to Construct the Five -way Corners Kepler's Solid

If you take the colored model of the Kepler solid in hand and look down at a five-fold corner, you see, reading from the right, five colors *1, 2, 3, 4, 5*. These corners can be built with the help of five steep blocks. If you want to make it comfortably, you need to make a saucer out of cardboard which fits over the Kepler solid like a cap.

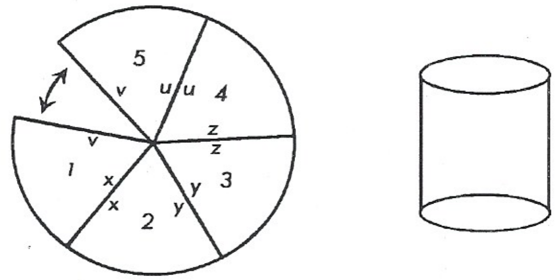

Figure 26

You also need a cardboard cylinder with both ends open to hold the saucers. A glass can be used instead. Put the saucer, having the surface *1*, *2*, ..., *5*, on the top opening of the cylinder and lay five steep blocks having the colors *1*, *2*, ... , *5* on the bottom on it. They will show the same colors on top; and they are joined according to the domino principle. The reader must now test what we say and build everything himself.

The appearance changes depending on which colors are used to join the five steep blocks.

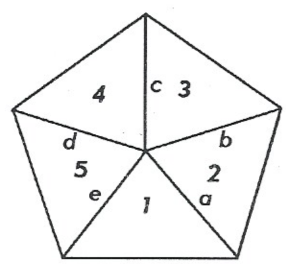

Figure 27

In figure 27 a general scheme is suggested showing what can occur. The block in area 1 of the saucer and the block in area 2 join together along the color *a*, the block on 2 and 3 with the color *b*, etc. Because the single blocks carry the color triads listed below.

1 e a   *2 a b*   *3 b c*   *4 c d*   *5 d e*

Thus the permutations of *a*,...,*e* submit to the following conditions:

a	is different from	*1*, *2*
b	is different from	*2*, *3*

c	is different from	*3, 4*
d	is different from	*4, 5*
e	is different from	*5, 1*.

This offers the following possibilities:

a	*b*	*c*	*d*	*e*
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

The five blocks make a cycle, so you must look at the columns with *a* and *e* at the top as being neighbors. You will notice by inspecting the table that each column has a number in common with both of its neighbors. These values are special when we have decided on the ordering *a*, *b*, *c*, *d*, *e*. Column *a* has a *4* in common with its neighbors *b* and *e*; *b* has a 5 in common with *a* and *c*; *c* has a *1*; *d* the number 2; and *e* the number 3. Because of this we will place the sequence *a*, *b*, *c*, *d*, right after *4*, 5, *1*, 2, 3. The five steep blocks, which we want to use for the production of the observed corner of the Kepler solid carry the following color triads:

1 3 4   *2 4 5*   *3 5 1*   *4 1 2*   *5 2 3*

If you number the five flat regions of the saucers *1*, *2*, *3*, *4*, *5* going counterclockwise, then each number lies opposite itself, that is united with the non-neighboring numbers of the triad.

Let us have *1*, *2*, *3*, *4*, *5* fall together with red, yellow, green, blue, white; the reader has to search the following triads to build the observed corner:

red	green	blue
yellow	blue	white
green	white	red
blue	red	yellow
white	yellow	green

From this the reader will be able to build the desired corners easily on the saucer. Observed from above it resembles a five pointed star with the color sequence red, yellow, green , blue, white, reading counterclockwise.

Around the edges, guided by the domino principle, you can fit in five stubby blocks. The third color will be determined by the opposite sheet of the just mentioned stars. For example, between the steep blocks

red, green, blue, and yellow, blue white
the stubby block green, white, blue, between
yellow, blue, white and green, white , red
blue, red, white, between
green, white, red and blue, red, yellow
white, yellow, red, between
blue, red, yellow and white, yellow, green
red, green, yellow, between
white, yellow, green and red, green, blue
yellow, blue, green

The colors red, yellow, green, blue, white will be cyclically switched at each step, that means each one is replaceable by the one following and the last one by the first one, thus red by yellow, yellow by green, green by blue, blue by white, white by red.

The first problem here has been the production of what is built up out of these five steep blocks and five stubby blocks. We built a type of cup. To tighten the model put a rubber band around the whole thing. The rubber band will go across ten vertical edges and the rhombi to which they belong. It takes some skill to put on the rubber band.

Filling the Cup

If you look into the cup, it now offers the possibility of using some of the remaining stubby blocks while respecting the domino rule. You might have already used the red, green, and blue stubby block, so that there are only two ways to continue with another stubby block. If you settle on red, yellow, blue no more stubby blocks can be fit in. If you really have done the construction work, you can check for yourself.

After filling in both stubby blocks – a red, green, blue one and a red, yellow, blue one – we see a hollow in the middle of the cup which invites the insertion of a red, blue, white steep block. On both sides of the inserted block are valleys in which a yellow, green, blue and also a red, yellow, green steep block fit in. The next step is the insertion of a stubby block. Indeed, you have a choice a between red, green, white one and a yellow, blue and white one. we will choose the red, green and white one. When this stubby block is added to your construction, you will see clearly a bed for the green, blue, white steep block and, after its insertion, a bed for the last steep block: red, yellow, white. The stubby blocks yellow, blue, white, and yellow, green, white can be accommodated without having to think about how to do it. This completes the construction of Kepler's solid.

Six rubber bands are needed to hold the blocks together because there are six families of parallel edges.

An Exact View of the Construction

If you pay attention to the way in which the twenty blocks that make up Kepler's solid come to the outside surface, you can recognize the following in your model:

Nine of the ten steep blocks each contribute a rhombus to the surface. One steep block lies hidden in the interior of Kepler's solid. Seven of the ten stubby blocks each provide three rhombi at the surface: The other three remain latent inside.

The twenty building stones of Kepler's solid have, altogether, $20 \times 6 = 120$ rhombi. Thirty of them supply the surface of the solid. The ninety others must, in pairs, make up the interior walls of the construction. Therefore each interior wall is doubled. There are thus a total of 45 inner walls which come apart onto the 20 pieces that make up the solid body. If you put a window in each rhombus that lies on the outside surface, the nine steep cells have only one window, the seven stubby cells have three windows. Sixteen cells would therefore take in light from outside. Four cells, one steep and three stubby, would not get any daylight.

The six bands around our building make a triangle around each three sided corner of Kepler's solid and a pentagon around each five sided corner. Disregarding the corner points within these triangles and pentagons, Kepler's solid is made up of 12 pentagons and 20 triangles. If we consider projecting the solid out from its center onto a circumscribed sphere, we obtain a partition of the sphere into 12 pentagons and 20 triangles the edges of which are all of equal length. They make great circles that intersect each other in thirty points.

You can easily create this partition on a rubber ball if you draw a great circle and then divide it into ten equal pieces. The ten pieces of this great circle will have equilateral triangles alternating up and down. The new sides of these triangles give us five more great circles. You really need only the triangles that point up. If you extend the left upper sides into the great circles of which they are a part, you will easily obtain a partition of the sphere. It is even easier to circumscribe circles around neighboring upper and lower triangles. This way you get ten of the twelve pentagons and without difficulty obtain the as yet missing parts of the partition. If you color the six great circles with different colors there arises a beautiful six colored model on the rubber ball. The following observation is of importance. You can arrange it so that each band lies alternately above and below the other at their junctions. The reader can test for himself that this rule can be realized without any inconsistency, if he is willing to construct for himself or to obtain such a colored ball. It hardly needs to be said that these Kepler balls are something new an offer a surprisingly pretty appearance.

Successive Views of the Kepler Ball

If you indicate the six colors which appear on the great circles of Kepler's ball with the numbers *1* through *6*, so that each of the 12 pentagons offers five numbers in cyclic order, then the pentagon on the opposite side has the same numbers taken cyclically in the reversed order. Each triangle shows a triad taken from *1*,..., *6*; while the opposite triangle has the same triad but in the reversed order. Therefore, on the whole ball you can read off six cycles of five numbers and ten cycles of three numbers each. On my model I find the following five-fold cycles

$$(\dagger) \qquad \begin{cases} 24356 & 4536 & 4625, \\ 12563, & 3246 & 2345. \end{cases}$$

and the three-fold cycles

$$(*) \qquad \begin{cases} 123, & 125, & 136 & 45, & 235 \\ 234 & 246 & 256 & 345, & 124. \end{cases}$$

Twenty triads can be created out of six numbers. Ten of these have been selected here. If you create the complement of a triad consisting of the other three numbers, you will find

$$(**) \qquad \begin{cases} 456, & 346, & 245 & 236 & 235 \\ 156, & 35, & 134, & 126, & 124, \end{cases}$$

thus the missing triads appear.

The ten triads that were selected form an *antipode-free system* (this expression is due to Arnold Kowalewski). Two triads which together exhaust all six numbers are said to be *antipodal.*

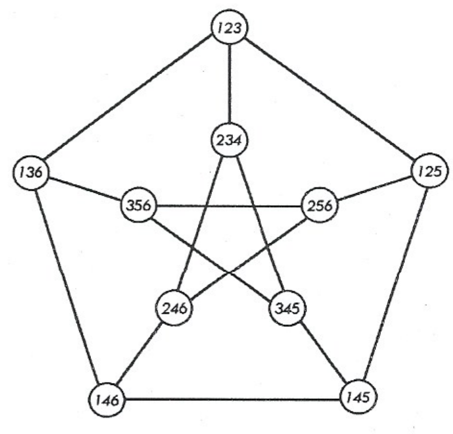

Figure 28

The ten distinguished triads in the list marked with a star above, has the following property which can also be established for (∗∗) as well. We note that each triad *a b c* includes the ordered pairs

a b, *a c*, and *b c*;

altogether there are thirty ordered pairs. These are all the ordered pairs that be created out of *1*,..., *6*, each written down twice. Because of this property we call (∗∗) a *Steiner triple system of the second order*. A Steiner triple system of the first order is also called simply a *Steiner triple system*. The Steiner triple systems will not be used here. One such system must consist of five triads within which all 15 ordered pairs made up of *1* to *6* are contained. If *1 2 3* is one of these triads, there would have to be another that contains the pair *1 4*. We can use *5* as a name for the new element that is different from *1*, *2*, *3*, and *4*. Besides *1 2 3* and *1 4 5* there cannot be triads missing so that would cause *1 6* to be omitted. But the third element cannot fall together with the numbers *2*, *3*, *4*, *5*. So we come to an impossibility. It is not possible to build a simple Steiner triple system out of *1* to *6*, but as the example (∗) shows there is a Steiner triple system of the second order.

In figure 28 you can see the triads of the list (∗) distributed into two pentagons whose corners are related using connecting lines. One is an ordinary pentagon and the other is a pentagonal star. From each triad there are pathways leading to three other triads, and indeed just to those triads that share an ordered pair in common.

The system of rounds (†) stands in a simple relation to the triad system (∗). Each round contains five triads which consist of one element and its two neighbors in the round. The entire six rounds deliver 30 such triads, and indeed the they are the triads (∗) written down three times each.

It is interesting to ask: how many different Kepler balls are there? Switch two elements in the list (∗), *1* and *2* for instance. At the same time switch the two elements that are combined into triads with *1* and *2*: In this case it is *4* and *5*, so this gives

1 2 5, *1* 2 3, 2 5 6, 2 3 *4*, 2 *4* 6,
1 *4* 5, *1* *4* 6, *1* 3 6, 3 *4* 5, 3 5 6.

Therefore, it is the triad system (∗), only in a different order. The transposition that switches two elements a and b is usually expressed by the symbol (ab). If a', b' are elements which in (∗) appear as an ordered pair embedded in some triad, then this system remains invariant under the switching elements (ab) and ($a'b'$). In all there are 15 such switchings, namely

(*1* 2) (3 5),	(*1* 3) (2 6),	(*1* *4*) (5 6),	(*1* 5) (2 *4*),	(*1* 6) (3 *4*),
	(2 3) (*1* *4*),	(2 4) (3 6),	(2 5) (*4* 6),	(2 6) (*4* 5),
		(3 *4*) (2 5),	(3 5) (*4* 6),	(3 6) (*1* 5),
			(*4* 5) (*1* 3),	(*4* 6) (*1* 2),
				(5 6) (2 3).

With the help of the switchings in the first row you can bring any of the elements 2, 3, *4*, 5, 6 into the first position in place of *1*, without altering the system (∗). When *1* is held fixed this brings to our attention the switchings

(2 *4*) (3 6), (2 6) (*4* 5), (3 4) (2 5), (3 5) (*4* 6), (5 6) (2 3).

With the help of these switchings you can steer any of the elements 2, 3, *4*, 5, 6 into the position of 2. Once *1* and 2 are fixed, there remains only the switching (3 5)(*4* 6); that is, you can exchange 3 and 5 in the triads they form with *1* and 2, and also exchange *4* and 6 without changing the system (∗).

Are there other switchings of the numbers 3, *4*, 5, 6 that leave the system (∗) unchanged? Since 3 and 5 and also *4* and 6 have to be permuted with each other, it only remains to check whether (3 5) and (*4* 6) separately will leave the system unchanged. Neither does. For example, (3 5) and (*4* 6) act on the triad *1* 3 6 to give *1* 5 6 or *1* 3 *4* respectively: These are triads from the system (∗∗). Apart from the identity, which leaves everything in place, there is only one switching of 3, *4*, 5, 6 that leaves the system (∗) unchanged, namely (3 5)(*4* 6). Altogether there are $1 \cdot 2 \cdot 3 \cdot 4 = 4! = 12$ different switchings that can be carried out on the numbers 3, *4*, 5, 6, so there are 12 systems that arise from by switching its elements. We have made use of a theorem of group theory that has been known for a long time.

You can check what has been done by thinking as follows: Along with the ordered pair *1*, *2*, the additional pair *3*, *4*, or *3*, *5*, or *3*, *6*, or *4*, *5* or *5*, *6*, will appear within a triad. That makes six possibilities. Each time there are two different systems that are transformed into each other when the elements that are in triads with *1* and *2* are switched with each other. For example the system (∗) changes into the following new system when *3* and *5* are switched:

1 2 5,	*1 2 3*,	<u>*1 5 6*</u>,	<u>*1 3 4*</u>,	*2 4 6*,
<u>*2 4 5*</u>,	*1 4 6*,	<u>*2 3 6*</u>,	*3 4 5*,	*3 5 6*.

The new triads are underlined.

The answer to our question therefore, is that 12 different Kepler balls can be created with six colors.

CHAPTER FOUR

Games Played with Thirty Little, Colored Men and Roots with Kepler's Solid

Upon reflection a lot can be said about Kepler's solid that we made out of twenty blocks and bandaged together with six colors in order to hold the building stones together. We already recognized that each band can be arranged in a handy way with crossings going alternately over and under. On each of the rhombic faces, the reader needs a model in hand, two bands cross, one on top and one beneath. Let us chose the colors

black, red, yellow, green, blue, white

for these bands. On each rhombus, one color is on top and one on the bottom. The top color is the color of the band that goes above and the bottom color is the color of the band that is beneath. If you name the colors *1, 2, 3, 4, 5, 6,* then each rhombus has a top number *a* and a bottom number *b*. This gives an ordered pair *a b*. thirty ordered pairs can be formed from the numbers *1* to *6*. Any of the six numbers can occupy the first position and after it is seated any of the remaining five can occupy the second position. These thirty ordered pairs of six objects are distributed on the thirty rhombic faces of Kepler's solid. On neighboring faces, that is, those sharing a common edge, there are pairs of the form *a b* and *c a*, in which *a*, *b*, and *c* represent three distinct terms of the sequence *1, 2, 3, 4, 5, 6*. The band that passes over the common edge of these two rhombi is *a*: it is on the top at one face and on the bottom at the other. The two ordered pairs have, as we see, an element in common but it takes the first position in one ordered pair and the second position in the other; this weakens the domino character of the neighborhood. You could call *a b* , *a c* or *b a* , *c a* a strong domino junction and *a b* , *c a* a weak domino junction. The thirty ordered pairs made out of *1, 2, 3, 4, 5, 6* are distributed over the rhombi of Kepler's solid so that neighboring pairs stand at a weak domino junction. This distribution will be given more attention. My brother, the founder of *systematic color theory,* an extraordinary deep theory with rich connections to practical questions, calls this distribution a *colonization*.

You can pose a colonization problem whose solution is found in the reasoning given above: the thirty ordered pairs formed from *1*, *2*, *3*, *4*, *5*, *6* have to be distributed on the faces of Kepler's solid so that neighboring pairs are connected according to the weak domino principle.

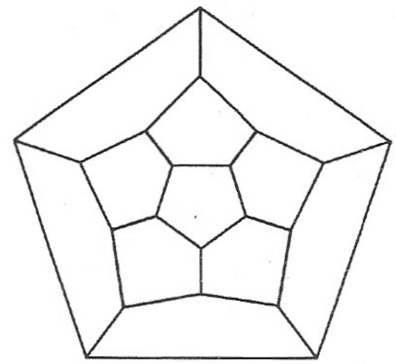

Figure 29

This colonization can be represented in a plane diagram because the resulting distortion does not change the nature of anything. Figure 29 shows a regular dodecahedron that has been pressed flat. the small pentagon lies on the big one like a sheet and the and the large pentagon is merged with the edges of the entire figure. If you were to blow this up as if it were an airtight bag, it would pop out as a figure in space resembling a regular dodecahedron. We know that Kepler's solid consists of the rhombic skirts that are shared alike by the regular dodecahedron and icosahedron alike. We can do this with a distorted dodecahedron just as well as a regular dodecahedron: but we have to be satisfied with representing the rhombi as quadrilaterals. Choose a point on each face of the distorted dodecahedron and connect it with the corners. When you take away the edges of the dodecahedron, there will be a distorted Kepler solid.

This construction is carried out in Figure 30. The dodecahedron edges are still visible as dashed lines. The point within the large pentagon that makes a basis for the figure has been thrown to infinity. We have to think of straight lines going to infinity from the corners of the large pentagon. It is though we were taking here a sphere to be a plane of infinite radius. The point at infinity is the point that is opposite the center point of the diagram.

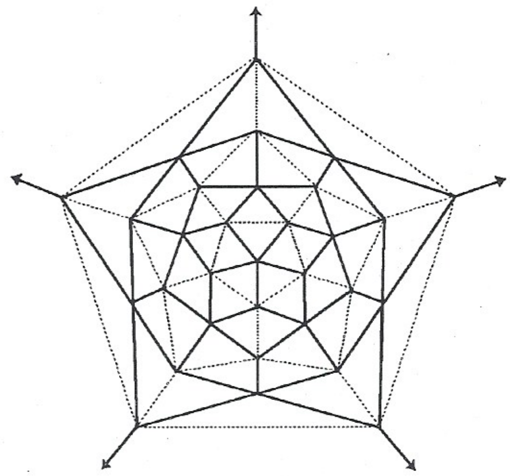

Figure 30

If you pay attention only to the dark lines and think away the dashed lines, you can see the plane divided into thirty quadrilateral areas, from which five lines go out to infinity. these thirty quadrilaterals will do for the colonization problem instead of the rhombi of Kepler's solid. In figure 31 you can see the ordered pairs taken from *1, 2, 3, 4, 5, 6*, the ordered pairs

*	1 2,	1 3,	1 4,	1 5,	1 6,
2 1,	*	2 3,	2 4,	2 5,	2 6,
3 1,	3 2,	*	3 4,	3 5,	3 6,
4 1,	4 2,	4 3,	*	4 5,	4 6,
5 1,	5 2,	5 3,	5 4,	*	5 6,
6 1,	6 2,	6 3,	6 4,	6 5,	*

placed into the thirty regions of the diagram according to the weak domino principle.

If you want to make a game out of this colonization problem, you replace the ordered pairs with little men: in the pair *a b*, *a* represents the color of his pants and *b* the color of his jacket. The thirty colored men are to be distributed on the regions of the game board so that two men in neighboring regions have a color in common on the same type of clothing. The common color makes a kind of neighborhood connection between them but it comes out in different ways. The pants color of one is on the jacket of another, so each little man, looking across his four boundries, can say with satisfaction that no neighbor is exactly like himself. If he has, for example, red pants and a blue jacket, then two neighbors have red jackets but not blue pants and the other two have blue pants but not red jackets. Every little man is special, different from his neighbors.

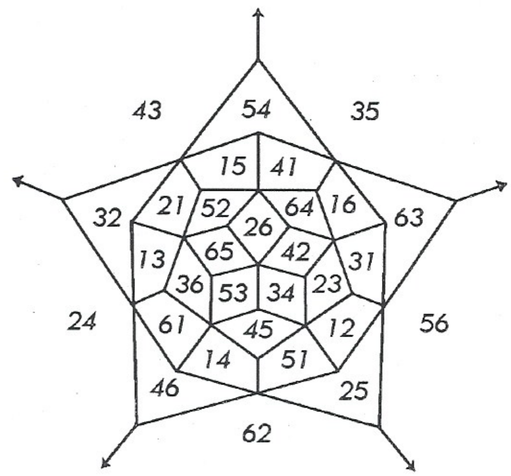

Figure 31

In the colonization game you can put, for example, two men that do not share a common color down onto regions that meet at a corner: then, using the principle of weak domino junction, go on with the colonization. In Figure 32 you can see one of these beginnings. Into the region marked with a star, that bounds the regions that have already been used, comes a diad *a b* which is related to both *1 2* and also *3 4* by a weak domino junction. To connect itself in the right way to *1 2*, it has to be either *a 1* or *2 a*. The diad *a 1* connected in the previously described way to to *3 4* when *a = 4* and *2* a is connected when *a*

= 3. Therefore the region ✱ must either be filled with *4 1* or *2* 3. If you choose *4 1*, then the diad *2 4* definately goes into the region ✱✱. If you were to choose 2 3, you will have to fill in ✱✱ with *3 1*. The same certainty will arise for filling in the region ✱✱✱. We shall leave it to the reader to finish the game board colony. That is the best way to get involved in the whole thing.

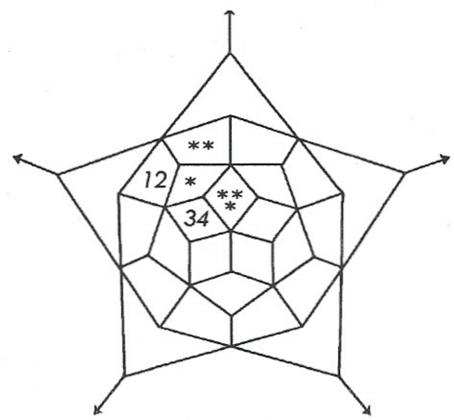

Figure 32

If you are a daring player you can choose more than a two-fold beginning originally and the see whether you can still play all the little colored men. Even if not all of them have been used, you still might play a large enough number of them that you are not embarassed to show it to someone. This number could be considered as your score for the game. To players can take turns trying their luck. Add up the scores compare the totals at the end.

Anyone who knows the theory can solve the colonization problem if nothing is showing in the beginning. You can solve it easily by picking out a ring of ten regions corresponding to one of the six bands around Kepler's solid. Each region of one of these rings contacts the ring of its neighbors along two opposite edges. In Figure 33 a ring is shown using boldfaced lines. You can begin to fill the regions of this ring with the diads *1 2*, *1 3*, *1 4*, *1 5*, *1 6*. On the opposite sides put *2 1*, *3 1*, *4 1*, *5 1*, *6 1*. The only thing that can go into the region the bounds *1* 2 and 5 *1*, for example, is the diad 2 5; only 3 5 can touch 5 *1* and *1* 3, etc. The regions bordering the ring can be settled in this way. Ten of the five-way corners will be missing a diad which is then forced by the demands of the weak domino junction rule with regard for the diads that have already been used. The reader can check this by continuing the columnsthat have been started in Figure 33.

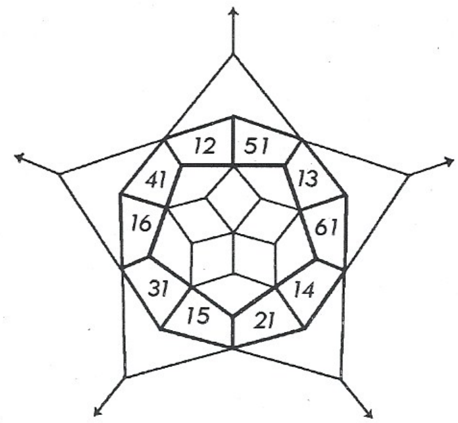

Figure 33

If you consider diads to be made of the numbers *0, 1, 2, 3, 4, 5* instead of *1, 2, 3, 4, 5, 6* and make the number $6A + B$ correspond to the diad $A\ B$, then the numbers that are obtained are all different. If would follow from

$$6A + B6 = A' + B'$$

that $$6(A - A') = B - B'$$

It would follow that $B - B'$, a difference of two numbers taken from the sequence *0, 1, 2, 3, 4, 5*, is a multiple of 6. That is possible only if $B - B' = 0$. It would follow immediately from this that $A - A' = 0$. As long as the diads $A\ B$ and $A'\ B'$ are distinct, it cannot be that the equations $A = A'$ and $B = B'$ hold simultaneously. So when the diads $A\ B$ are substituted into the expression $6A + B$, thirty different numbers are obtained. The smallest is 60 + 1 = 1: The largest is 65 + 4 = 34. The numbers of the form $6x + x$, that is 7, 14, 21, 28, the multiples of 7, are not included. The following 30 numbers remain:

1,	2,	3,	4,	5,	6,
8,	9,	10,	11,	12,	13,
15,	16,	17,	18,	19,	20,
22,	23,	24,	25,	26,	27,
29,	30,	31,	32,	33,	34.

If you have a Kepler solid that is not colored and ordered diads made up of numbers *0, 1, 2, 3, 4, 5* are spread out over the thirty rhombi, then by substituting the number $6A + B$ for the diad $A\ B$, the thirty numbers in the above table will be marked onto the rhombi. Kepler solids of this kind can be used as dice. To actually do that, they would have to be made out of wood or porcelin. This solid is much closer to a sphere than a cube is. It rolls easily across a table. However it stops, there is a number showing on the top face which can bbe read as the outcome of the throw. You could have twelve such dice numbered differently corresponding to the twelve different Kepler balls. They originate from the sequence given in the preceeding trable, and eventually will coincide with it. There are many possibilities here for working with the game rules, but we do not want to go into that now. There are also interesting problems for probability calculations.

Chapter Five

On the Rhombic Dodecahedron

The rhombic dodecahedron has, as we have learned, the same rhombic skirt as the cube and octahedron. You can also see it as an exception to solid bodies in general, we will now turn to its construction.

To get involved better into the nature of the situation, we want to treat a similar construction. Consider a parallelopiped, or as we prefer to say a *block*, lying in space. From a corner of this block go three line segments AB, AC, and AD. If we put them together in pairs like forces in the construction of the parallelogram of forces, that is, each line segment is brought to the ends of the others by a parallel translation, giving the six line segments BC_1, BD_1, CD_1, CB_1, DB_1, DC_1. We slide the original line segments to their endpoints as well an obtain in this way the three edges B_1A_1, C_1A_1, D_1A_1. In this way all twelve edges of the block are derived from the three basic line segments AB, AC, AD. If you project the block onto a plane by a parallel projection (in Figure 34 it is the plane ABD) there arises an image of the block in the plane.

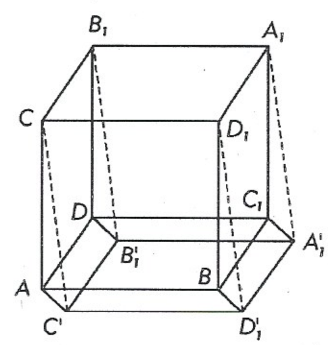

Figure 34

In this plane image everything is derived from the three basic line segments AB, AC', AD in exactly the same way as in the spatial construction. If you leave out the line segments going from A and A_1', there remains the hexagon $BC_1DB_1'CD_1'B$, the projection of the hexagon $BC_1DB_1CD_1B$, consisting of those six edges of the block that do not go to either of the two opposite corners A and A_1.

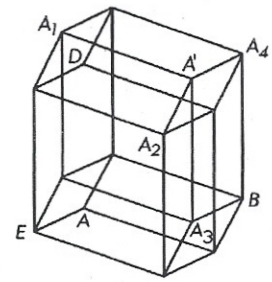

Figure 35

Using this construction we can design, in our ordinary space, the image of a four dimensional block in which we lay down four basic segments AB, AC, AD, AE and take them together in pairs, triples, and then four at a time as shown in Figure 35. You can carry out the construction so that you get a block out of the triples

∗,	AC,	AD,	AE,
AB,	∗,	AD,	AE,
AB,	AC,	∗,	AE,
AB,	AC,	AD,	∗

you will obtain a block and then put together all four line segments. This means that the four points A_1, A_2, A_3, A_4 that are all opposite to A in some block translate to give the missing edges. These line segments all meet at the point A'.

In Figure 36 we see a cube whose center A is connected by segments AB, AC, AD, to four of the cube corners. These corners are so chosen that no two have an edge in common. These four named segments are obviously of the same length and between any two of them they form the angle 2α such that $\tan \alpha = \sqrt{2}$. You indicate this, it reminds us of an irminsul, by the very important regular figure of four legs joined together at their ends.

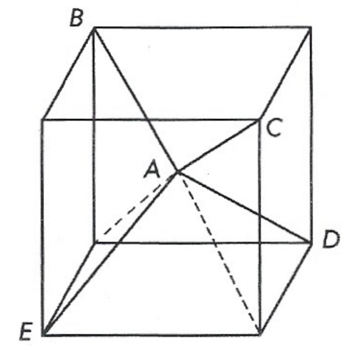

Figure 36

If you use these four legs for the construction given above, then the points A and A' will coincide (as in Figure 37).

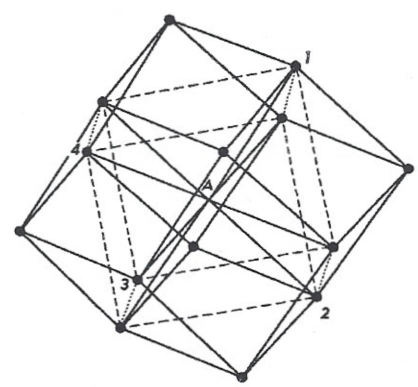

Figure 37

If you make a change in Figure 35 by leaving out the lies that come from the points A and A' ,then you have a parallelogram dodecahedron in front of you, so the rhombic skirts of the cube are a special case. The parallelogram dodecahedron arises from a four dimensional block through removing two opposite corners with their attached edges and then carrying out a parallel projection onto one of the infinitely many three dimensional spaces that play the same roll in four dimensions as planes do in three dimensions. Therefore the parallelogram dodecahedron is the spatial analog of the planar hexagon.

You see the parallelogram dodecahedron in the right light when you think of it in connection with a four dimensional block. We can see it as having its origin in four dimensional space , from a four dimensional block whose opposite corners have been removed with their attached edges. The four dimensional block has sixteen corners. Four edges go out from each corner, uniting in pairs and making six parallelograms. Altogether their are 16·6 parallelograms. Each of them will be counted four times however, because they have four corners. So the total number of these parallelograms in a four dimensional block is $\frac{16 \cdot 6}{4}$ or 24. We have pressed two opposite corners with their attached edges down underneath it all, so the six parallelograms that belong to them will be removed too. The twelve parallelograms that remain can be seen in the parallelogram dodecahedron.

The Source of the Rhombic Dodecahedron in Four Dimensional Space

The four dimensional origin of the rhombic dodecahedron can be seen by envisioning a cube in four dimensional space and then removing two opposite corners along with their attached edges. Only twelve of the 24 squares of this cube will remain. They present us with the four dimensional origin of the rhombic dodecahedron. The rhombi are congruent squares in this case. If an inhabitant of four dimensional space looks at this structure from a distance, along the line that connects the two opposite corners that were removed, he sees the rhombic dodecahedron as a three dimensional figure, thus in an entirely different way from the way that we see it. This viewer experiences, so to speak, something similar to what we experience when we look at a cube from a distant point along the line connecting two of its opposite corners, with the edges that attach to those corners made invisible. The picture that this offers to us is that of a regular hexagon. It is the projection of the remaining six cube edges onto a plane vertical to our line of sight.

Since we cannot look into a four dimensional space as we do our usual space, we need to depend on the analytical methods of Descartes for the treatment of geometrical questions in four dimensions. It is like Braille which offers a substitute for vision. In ordinary space a point is indicated by three Cartesian coordinates x, y, z in connection with three perpendicular axes. If you want to fix a specific cube, you have only to give the coordinate triples of its eight corners. If, for example, the coordinate triples

0, 0, 0	1, 0, 0	0, 1, 0	0, 0, 1
0, 1, 1	1, 0, 1	1, 1, 0	1, 1, 1

are written down, then you have to do with a cube that is supported by the three positive axes and has edges of length 1. The corresponding body in four dimensional space, where there are four rather than tree coordinates, has the corners:

0, 0, 0, 0
1, 0, 0, 0   0, 1, 0, 0   0, 0, 1, 0   0, 0, 0, 1
1, 1, 0, 0   1, 0, 1, 0   1, 0, 0, 1
0, 0, 1, 1   0, 1, 0, 1   0, 1, 1, 0
0, 1, 1, 1   1, 0, 1, 1   1, 1, 0, 1   1, 1, 1, 0
1, 1, 1, 1.

If you now remove the opposite corners 0, 0, 0, 0 and 1, 1, 1, 1, there remain the corners of the 12 squares that build up the structure which we call the four dimensional source of the rhombic dodecahedron. The endpoints of an edge always have coordinate quadruples that differ in only one position, the other terms agreeing. So the corners of a rhombic dodecahedron in our space can be given the 14 labels

(*)

1, 0, 0, 0   0, 1, 0, 0   0, 0, 1, 0   0, 0, 0, 1
1, 1, 0, 0   1, 0, 1, 0   1, 0, 0, 1
0, 0, 1, 1   0, 1, 0, 1   0, 1, 1, 0
0, 1, 1, 1   1, 0, 1, 1   1, 1, 0, 1   1, 1, 1, 0

Here the endpoints of each edge carry labels that differ only in one term, thus as little as possible, or, as we like to say, they stand in a strong domino junction.

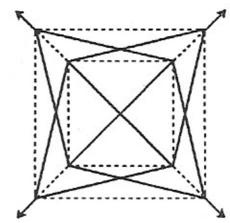

Figure 38

Figure 38 shows a plane representation of a rhombic dodecahedron that comes out of the cube that is shown in dashes. The midpoint of the big square is thrown to infinity, understood as the point toward which all four of the rays in the figure are heading. The 14 corners here need to given the labels (∗) with the strong domino principle in control along each edge.

Whoever knows the source of this net in four dimensional space will be able to solve the problem without difficulty. The edges of the original object, which are all edges of a four dimensional cube, fall into four classes, each made up of six segments parallel to one of the four coordinate axes. The same grouping applies to the lines in the net of Figure 38. If you realize that in the original object, edges that go out from one corner belong to different classes and on the opposite point to the same class, then it is easy to write down numbers *1, 2, 3, 4* representing these classes in Figure 38. You would start by numbering the lines going out from the center of the picture somehow with the numbers *1, 2, 3, 4* and as you proceed with the numbering you go along with the rule that squares that lie opposite get the same class numbers. Figure 39 shows a completed numbering of this kind.

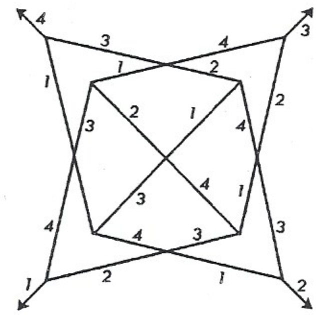

Figure 39

Now you have to think that running along an edge of class a, that is at the transition from one end to another, only the a^{th} term of the corner label will change, while the other terms remain the same. If you picture this to yourself, you can say that the labels 1, 0, 0, 0 and 0, 1, 1, 1 cannot possibly stand on corners linked by a line of class *1*. The other end of such a line would have to be either 0, 0, 0, 0 or 1, 1, 1, 1. But these labels are not available. Likewise, 0, 1, 0, 0 and 1, 0, 1, 1 cannot go on corners that emit a number *2* line. The corresponding thing holds for 0, 0, 1, 0 and 1, 1, 0, 1, as well as 0, 0, 0, 1 and 1, 1, 1, 0. The eight labels that have been enumerated, and thus also the eight intersections in question, emit only three lines.

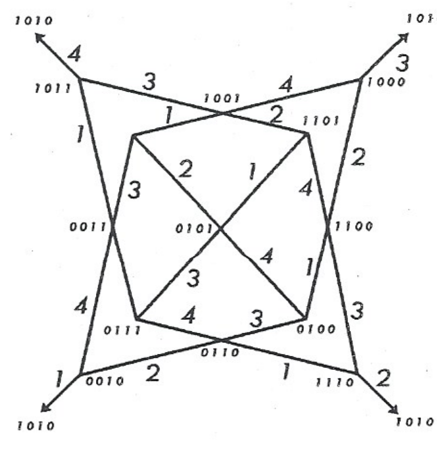

Figure 40

With this remark in hand we may partition the labels without any difficulty, as the reader can test for himself (Figure 40). If you stow away 1, 0, 0, 0 and 0, 1, 1, 1, as can be done in two different ways, it puts a further constraint on the labels because the class number of the edges always specify which term must be changed along such a line.

When the class numbers of the lines are not given, it is not so easy to stow away the labels especially for someone who doesn't know the secret of these net diagrams. To make a game out of it you replace the four termed sequences with little, white towers that have four bands going around them that are either black or red. For example, instead of the label 1, 0, 1, 0 there is a tower having the colors red, black, red, black going in order from the top to the bottom (Figure 41). Now you start occupying the thirteen circles in Figure 42 according to the strong domino principle so that the towers that are connected by lines differ on only one of their rings. When thirteen positions are occupied, there is one tower left over.

Figure 41

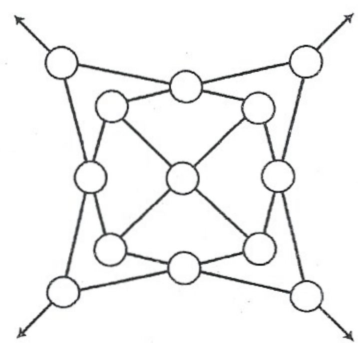

Figure 42

You can set up the game so that this king without a country is taken out of the game right in the beginning. If you choose the wrong one, you can't get the towers stowed away. The tower that is excluded must have two red bands and two black bands. That is the big secret. The game loses its appeal when people know that. Pure pleasure can only be found by a childish soul who is removed from the theory. When the towers cannot all be stored, you can still give points for correctly played towers.

Four Bands Around the Rhombic Dodecahedron

The rhombic dodecahedron is, as we have often pointed out, a parallel projection of a four dimensional original into a three dimensional space. Parallel lines remain parallel in the projection. From this it follows that the rhombic dodecahedron in our space has four classes of edges each of which consists of six parallel segments in the original. You can verify it with one glance at a rhombic dodecahedron, it shows in Figure 37 too.

Now we want to put two altitudes through the middle point of each rhombus perpendicular to the sides (Figure 43) and give these altitudes the same class number as the sides to which they belong. Six of these altitudes carry the number *1*, six carry the number 2, and so on. The altitudes with the number *a* stuck on them make a band around the rhombic dodecahedron; and an entire class of edges, the class *a*, goes across them.

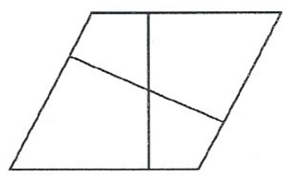

Figure 43

You can substitute four colors, red, yellow, green, and blue for instance, and give the bands a suitable width. If you color the whole rhombic dodecahedron black, it produces a model that makes a strongly aesthetic impression. Just as with the bands that surround Kepler's solid, you can arrange the bands so that they pass alternately over and under each other at their six junctions. On each of the twelve rhombi we have a top band *a* and a bottom band *b*. Let us return to using the numbers 1,2, 3, 4 and write the number of the top band in the first position and the number of the lower band in the second position, thus making ordered pairs out of the numbers 1, 2, 3, 4. These ordered pairs

*,	12,	13,	14,
21,	*,	23,	24,
31,	32,	*,	34,
41,	42,	43,	*

are spread over the twelve faces of the rhombic dodecahedron in such a way that the pairs that stand on neighboring faces always have an element in common, but it is in the first position of one pair and in the second position of the other. We call this relation, which also comes to us in the Kepler solid, the *weak domino junction.* If you want to carry out a colonization of the surfaces of the rhombic dodecahedron using ordered pairs with the elements *1*, *2*, *3*, *4* according to the weak domino principle, you can start by putting the pair *1 2* on any of the fields. The adjoining pairs *2 3*, *2 4*, *3 1*, *4 1* will be distributed among the four neighboring fields. These four fields can be seen to have equal status by rotating and reflecting the rhombic dodecahedron.

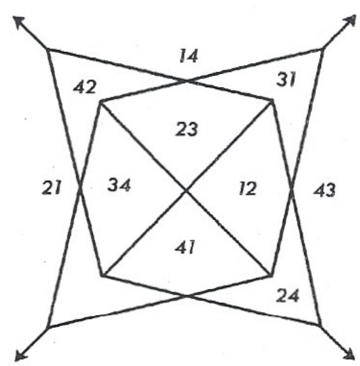

Figure 44

So the pair *2 3* could go on any of these four faces. Once that is done the positions of the remaining three are uniquely determined. This shows how the course of further colonization is forced. The result can be seen in Figure 44. Allowing for rotations, the whole thing has two colonizations of the desired kind: Figure 44 and its mirror image. As with Kepler's solid, you can make dice by replacing the ordered pairs on the faces of the rhombic dodecahedron with numbers. For this purpose it is advisable to use the numbers *0, 1, 2, 3* instead of *1, 2, 3, 4* and change the pair *a b* into the number $4a + b$. In place of the pairs

*,	01,	02,	03,
10,	*,	12,	13,
20,	21,	*,	23,
30,	31,	32,	*

there stands the numbers

*,	1,	2,	3,
4,	*,	6,	7,
8,	9,	*,	11,
12,	13,	14,	*

You will obtain two dice this way because there are two colonizations, not counting rotations. Because there are twice as many faces as there are on the dice of the usual kind, games played with rhombic dodecahedral dice offer more excitement.

If you project the four bands which we put around the rhombic dodecahedron outward from the center of the solid onto a circumscribed sphere so that four band-like great circles arise. They divide the sphere into six quadrilaterals and eight triangles. You may paint these bands on the sphere in such a way that each of them goes alternately over and under at the six junctions that it has with the other three. The actual construction, on rubber balls for instance, will not present the reader with difficulties. The two four color balls that are obtained will give all the more satisfaction because they cannot be bought anywhere.

The fourteen fields of such a sphere partitioned into six quadrilaterals and eight triangles correspond to the corners of a rhombic dodecahedron. Thus, the previously considered colonization of the corners of the solid with four termed labels consisting of zeroes and ones allows us to see the partitioning of the sphere into triangles and quadrilaterals as a field colonization as well.

Construction of the Rhombic Dodecahedron out of Four Blocks

If you think back on the construction of the rhombic dodecahedron out of regular four legged pieces and glance again at Figure 37, you will see four blocks out of which the solid rhombic dodecahedron can be made. Any three of the four legs makes a solid corner into which one of the four blocks can fit. The reader may build these four blocks out of cardboard. They are bordered by three rhombi whose diagonals are in the ratio $1{:}\sqrt{2}$. The blocks that are involved are clearly the stubby ones.

If you number the legs of one of these four legged pieces with *1, 2, 3, 4,* you can number the four blocks with the triads

2, 3, 4; 1, 3, 4; 1, 2, 4; 1, 2, 3.

the block 2, 3, 4 fits into the solid corner with the legs 2, 3, 4 so that three of its edges lie along the just mentioned legs. If you color the legs of the four legged piece, using for example red, yellow, green, and blue, then you can color red the edges of the block that are parallel to a red leg, and likewise the edges that are parallel to a yellow edge can be colored yellow and so on. carry out this coloring of edges so that the colors shows on both of the adjoining faces without making too small a colored strip. Three edge colors are used on each block. Around the block with red, yellow, and green going out for example, you will next paint the edge stripes that go out from an obtuse corner where three obtuse angles of the rhombi meet. It does not matter in which order you take the colors because, when you color according to the rule, the colors are cyclically reversed on the opposite corner. The blocks look very beautiful when you take black as the background color, or glue on black matte paper.

Joining the blocks together to make a rhombic dodecahedron is easy because you know, for example, that the red, green yellow block must rest itself on the red, yellow, green leg of the four legged piece, which, by the way, does not need to really exist. All four blocks match with four legged pieces at obtuse angles. The reader should convince himself how easy it is to join the four blocks together. To keep the construction together wrap a rubber band around each set of six parallel edges. These four rubber bands cause the previously mentioned division of the surface into eight triangles and six quadrilaterals stand out clearly. It is very beautiful if you use rubber bands that are colored like the edges that they cross.

NOTES

CHAPTER 5

irminsul see Hans Gsänger Mysteriensttten der Menschheit: Die Externstein verlag die Kommenden Frieburg , 1964
the irminsul is a tree of life symbol found in northern germany with runes. It seems to be a mixture of extra-Roman Christianity along with pagan elements. Odin was tied to a tree for nine days by the giants.